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ABSTRACT

This book, written by classroom teachers, introduces the application of secondary school mathematics to space exploration, and is intended to unify science and mathematics. In early chapters geometric concepts are used with general concepts of space and rough approximations of space measurements. Later, these concepts are refined to include the use of coordinates and measurements in space exploration. The first three chapters may be used as supplementary material to challenge students with space applications, as well as resource materials for the teachers. Chapters 4 and 5 provide basic concepts of science and mathematics needed to understand space travel. These concepts may supplement regular science computers in treating meaningful problems. (RR)

F-NASA

MATHEMATICS IN SPACE SCIENCE

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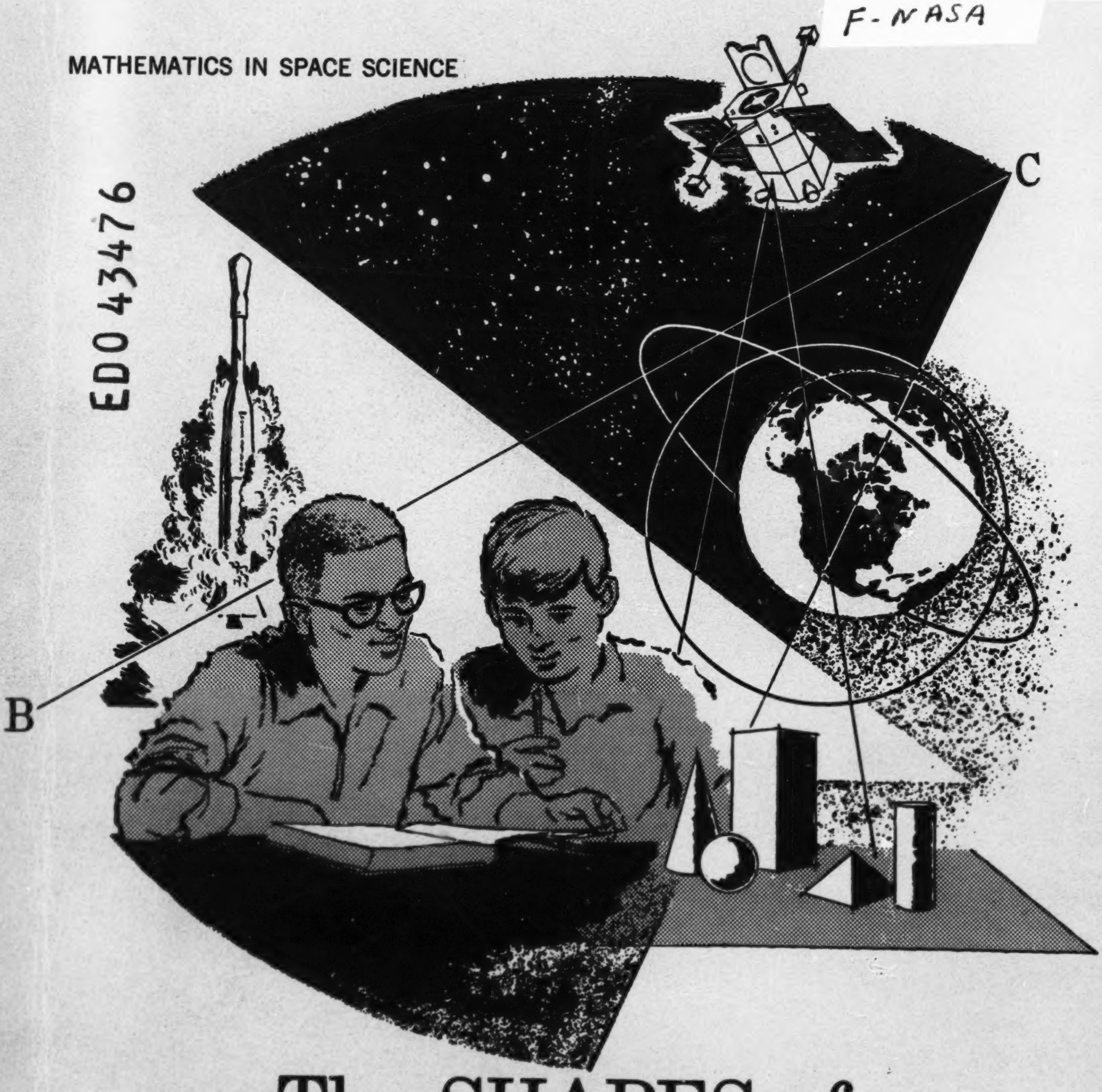

The SHAPES of TOMORROW

A supplement in space oriented geometry for secondary levels

SE 007 523

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The SHAPES of TOMORROW

A supplement in space oriented geometry for secondary levels

Prepared from materials furnished by the National Aeronautics and Space Administration in cooperation with the United States Office of Education by a Committee on Space Science Oriented Mathematics

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FOREWORD

This book introduces the application of secondary school mathematics to the exploration of space. It provides a unification of science and mathematics. This is not a textbook, although there are exercises to provide the reader with opportunities to test and extend his understanding. An inquiring mind, rather than a specified grade level in school, is a prerequisite for an enjoyable exploration of the ideas in this book.

While various treatments of this material may be found elsewhere and some elementary concepts are included, these are presented here in an easily accessible format, for supplementary classroom uses. There has been a deliberate effort to include enough familiar topics to make a meaningful transition from previous experiences to new concepts.

As the reader progresses through this book he encounters a spiral development of ideas. Elementary geometric concepts are introduced in Chapter 1 for readers who have not formally studied geometry. These geometric concepts are used with general concepts of space and rough approximations of space measurements. Then, in later chapters geometric concepts, space concepts and measurements are gradually refined as the maturity of the reader increases. In Chapter 2 the use of coordinates in the study of space is explored. In Chapter 3 the uses of measurements in our explorations of space are described. These first three chapters may be used effectively

to supplement the usual materials at any one of several secondary-school grade levels,

to challenge students with space applications, and as source materials for teachers (and writers) as they strive to capture the imagination of their students.

Chapters 4 and 5 provide basic concepts of science and mathematics that are needed to understand space travel. These concepts are appropriate as supplements for either science or mathematics courses. Chapter 6 illustrates the effective manner in which electronic computers may be used to treat meaningful problems.

The authors of this material are experienced classroom teachers who visited the NASA/Goddard Space Flight Center, consulted personally with many of its scientists and engineers, and studied important aspects of the U.S. space program. This book is their effort to share these experiences by providing other teachers with enrichment materials for use in motivating their students.

Bruce E. Meserve
University of Vermont

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About This Book

It is logical to presume that major achievements in the exploration of space rest with the youth of today and with the education they receive. It is therefore our sincere concern that the space program be conducted in close cooperation with our Nation's educational institutions. Understandably, young people throughout the land are fascinated by the era of space travel. Teachers and students have long demonstrated their eagerness to relate the subjects they study to the space program. However, frequently there appears to be a lack of available reference material suitable for classroom use.

In view of the continuing demand for such material, especially for elementary and secondary levels, the NASA/Goddard Space Flight Center, in cooperation with the U.S. Office of Education, has initiated a program of summer workshops to develop space-oriented mathematics supplements. The program is directed by a Committee on Space-Oriented Mathematics consisting of Dr. Patricia Spross, Specialist in Mathematics, U.S. Office of Education, and Mr. Alfred Rosenthal, of the NASA/Goddard Space Flight Center. Mr. Elva Bailey, Goddard Educational Programs Officer, serves as materials coordinator.

This publication, the third in the series, focuses on the use of geometric concepts. It was prepared during a summer workshop held at the University of Vermont, Burlington, Vermont, under the direction of Dr. Bruce E. Meserve, Department of Mathematics, College of Technology.

Overall guidance and direction for this project has been provided by the Office of Educational Programs, National Aeronautics and Space Administration.

MICHAEL J. VACCARO
Chairman, Committee on
Space-Oriented Mathematics

Chapter 1

DESCRIBING THE SHAPES OF THINGS

by

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DESCRIBING THE SHAPES OF THINGS

1–1 Shapes on Earth

Think of the part of our Earth over which you may have walked. Except for hills and valleys, mountains and waterways, it can be considered flat. Walking is a very simple way of getting from one place to another. And as you look toward the horizon, it appears that there is an edge of Earth and perhaps the sky comes down to meet it. It is not surprising, therefore, that before the time of Columbus most people believed that their Earth was flat, and that sooner or later, if they traveled far enough, they would come to the edge and drop off! Their World was the one they could see by walking around upon it.

As time passed, men invented improved ways of travel by using wheels on land and sailboats on water. Travel with wheels, however, required the construction of roads. The roads on water were already there: It was natural that men turned to boats for traveling greater distances. So long as they kept land in sight they could start and stop when they pleased, could tell where they were, and felt sure that they would not "drop off."

Man's curiosity and courage impelled him to develop instruments that would enable him to travel on the water to find what was beyond the sight of land. The stars had long been guides for travel at night. The invention of the compass made it possible to chart a course both day and night. The art of navigation reached such a refinement that men could travel farther from land than they had ever dared to venture before. Finally when Columbus set out many people were certain that both men and ships would be lost over the edge of the World forever.

Figure 1–1

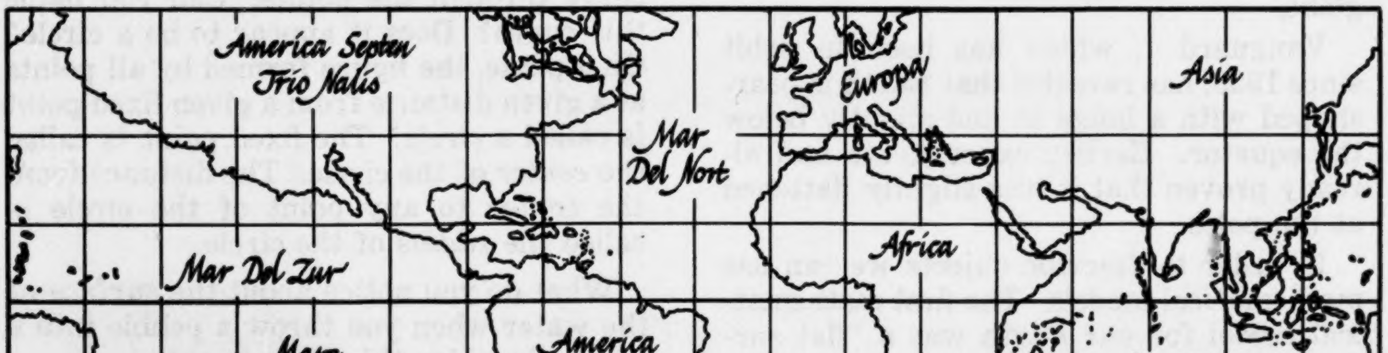


Figure 1–2

Although Columbus did not prove the World to be round, nor find a new route to India, he did find a "New World" and the maps began to be changed.

However, man has never appeared to be satisfied in his search for knowledge and explorations have continued to tell us more about our Earth and the space in which it revolves.

You are just beginning to study about those whose contributions in science and mathematics make travel in space possible. One of the early pioneers in space travel was Robert H. Goddard whose scientific curiosity led him to work on the development of an efficient means for space transportation: the rocket engine. It is interesting to note that the efforts of pioneers are not always appreciated. Robert Goddard was forced to move from New England to the desert of New Mexico in order to carry on his work because the noise of the rockets bothered his neighbors. He lives now in our memory as one of the greatest of the early space explorers. The Goddard Space Flight Center in Greenbelt, Maryland is named for him.

The ability to get into space has given us a better look at our Earth. Maps are becoming more accurate, and we have been able to describe more precisely the true shape of the planet on which we live. Like Columbus we have thought it was a round globe.

Vanguard I, which has been in orbit since 1958, has revealed that Earth is pear-shaped with a bulge at and slightly below the equator. Earlier experiments had already proven that it was slightly flattened at the poles.

In order to describe objects we can use mathematical models. The first mathematical model for our Earth was a "flat surface." What exactly do we mean by a "flat surface"? First we need to think of a *line* as being composed of an infinite set of points and having only one dimension—length. When we speak of a line in this book we will refer to a straight line unless otherwise stated. If two points of a line are on a flat surface, then every point of the line is on that surface; we say that the line is on the surface. Any two points determine a line. If every point of each line that is determined by points of a surface also lies on the surface, then the surface is flat and is called a *plane*. A table top is an example of a plane surface. We consider a line to be of infinite length and a plane to extend infinitely far in all directions. A plane like a line is a collection of points. It does not have thickness.

1–1 Exercises

Shapes on Earth

1. Name several examples of plane surfaces.
2. On a plane surface how many straight lines could be drawn through two points?

1–2 Earth's Atmosphere

On Earth you are surrounded with something you cannot see—air. Many questions about air have been studied. How far into space does it extend? Does it extend to the moon? Does it hold us "down" here? To make a mathematical model describing the earth's atmosphere we need first to think about circles.

Let's try an experiment. Take a sheet of paper and with your pencil mark a point P on it. Select as many points of the paper as you can that are one inch from P. Label some of these points A, B, C, . . . Draw a curve through the points. Can you name this curve? Does it appear to be a circle? On a plane, the figure formed by all points at a given distance from a given fixed point is called a *circle*. The fixed point is called the *center* of the circle. The distance from the center to any point of the circle is called the *radius* of the circle.

What do you notice about the surface of the water when you throw a pebble into a lake? Consider this experiment.

Draw a circle with a ½" radius. Locate a point 1" from the center of the circle. Find the path or paths that all points 1" from the center of the given circle would form. Find the path that all points located 2" from the center of the given circle would form. Try finding all the points ¼" from the center of the given circle and the path they would form.

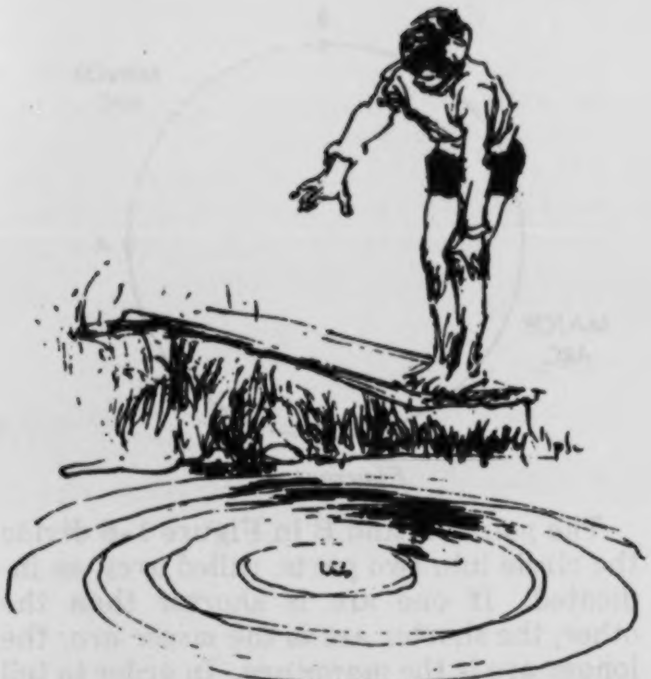


Figure 1–3

Do you find that some points form a path inside the given circle? Some outside? You should find that all points which were 1" and 2" respectively from the center of the given circle are *outside* the circle. The points ¼" from the center of the given circle are *inside* the circle. Where do the points ¾" from the center of the given circle lie? In each case the points were points of circles having the center of the given circle as center. Any two circles with the same center on a plane are *concentric circles.* The waves from a pebble dropped into a lake often appear as concentric circles. (Exercises 1 and 2 are concerned with circles).

On portions of the surface of our Earth we are concerned with directions such as north, east, south, and west (as on a plane), but we can also look up into space. We live in a three-dimensional world. A plane is only two-dimensional. In our three-dimensional world the set of all points at a given distance (radius) from a fixed point (center) is a *sphere.* Think of several objects that are shaped like spheres. Are your examples exactly spherical in shape? Can you make a definition to describe concentric spheres? Compare your definition with the definition of concentric circles.

For many years we have thought of the surface of Earth as approximately spherical. We think of the different layers of Earth's atmosphere as also bounded by surfaces that are approximately spheres. It is important to know about these layers for in each we can study their effects on Earth and life. These layers "wrap" themselves around the Earth like an orange with many layers of skin,each layer blends into the adjoining areas.

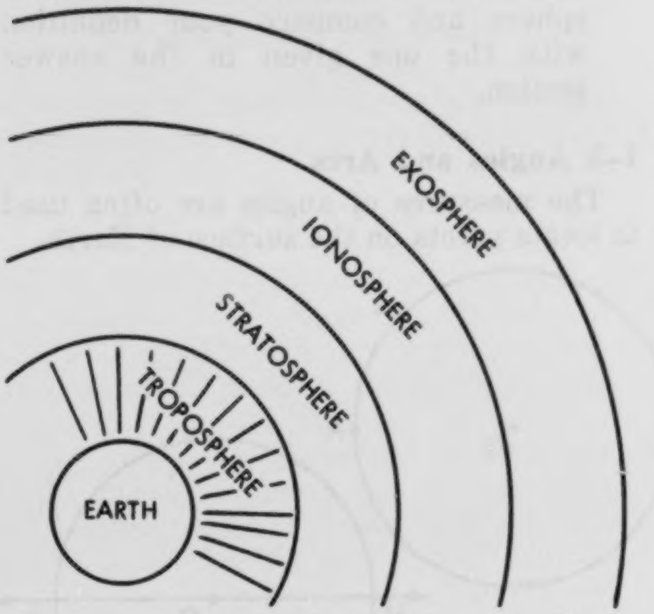


Figure 1–4

If we take a trip into space we find that the first layer of Earth's atmosphere (*the troposphere*) is five to eight or more miles thick with considerable variations. Our weather conditions occur primarily in the troposphere. There is also decreasing temperature as one leaves the surface of the earth and there are many "up drafts" movements of air.

After we travel through the troposphere we enter the *stratosphere* which extends to about 50 miles above the surface of the Earth.

The third layer (sometimes considered as several layers) is referred to as the *ionosphere,* extends to about 400 miles above Earth, and contains electric particles called ions.

Beyond the ionosphere is the *exosphere* and outer space.

1–2 Exercises

Earth's Atmosphere

1. Trace around a coin that has been put in several different places on a piece of

paper. Do you think the circles you have drawn are about the same size? We call these congruent circles. Why do you think their radii are called congruent?

2. Describe several examples of concentric circles.
3. Describe several spherical objects.
4. Make a definition for the radius of a sphere and compare your definition with the one given in the answer section.

1–3 Angles and Arcs

The measures of angles are often used to locate points on the surface of Earth.

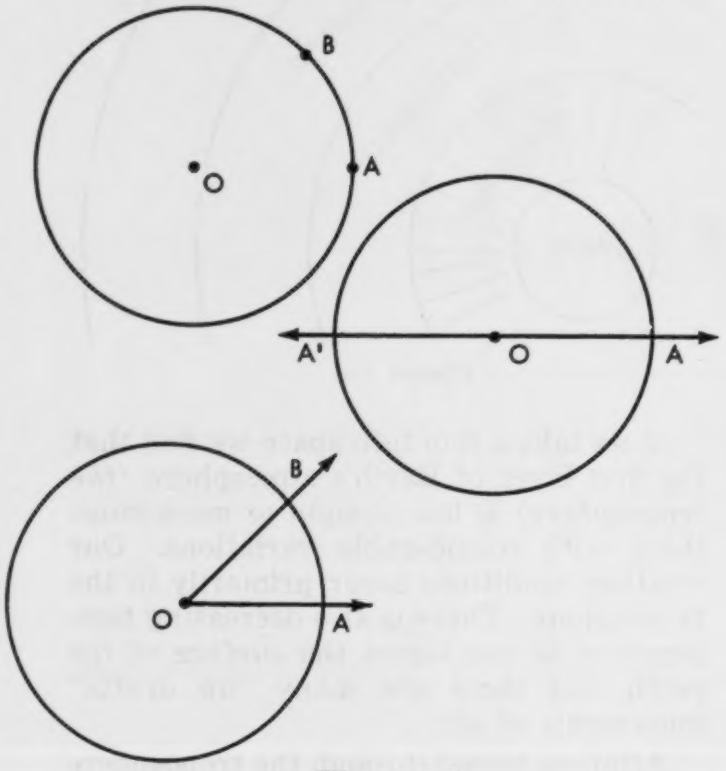


Figure 1–5

Consider these models of a circle with center O and points A and B. There is a line OA that intersects the circle in points A and A′. The point O and the points of the line that you would traverse in traveling from O through A and continuing indefinitely forms the *ray OA*. There is also a ray OB. The two rays OA and OB form an *angle* (∠AOB). The rays OA and OB are *sides* of ∠AOB. The point O that is common to both rays is the *vertex* of ∠AOB. Since the vertex of ∠AOB is also the center of the circle, ∠AOB is a *central angle*.

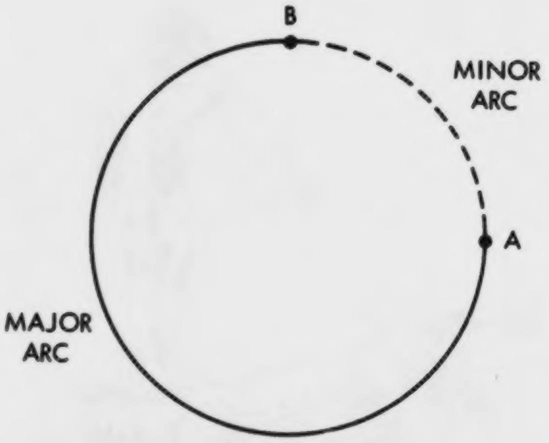


Figure 1–6

The points A and B in Figure 1–6 divide the circle into two parts, called *arcs*, as indicated. If one arc is shorter than the other, the shorter arc is the *minor arc;* the longer arc is the *major arc.* In order to tell which arc is the longer, we need to have some way of measuring arcs. Measurements are explained in detail in Chapter 3.

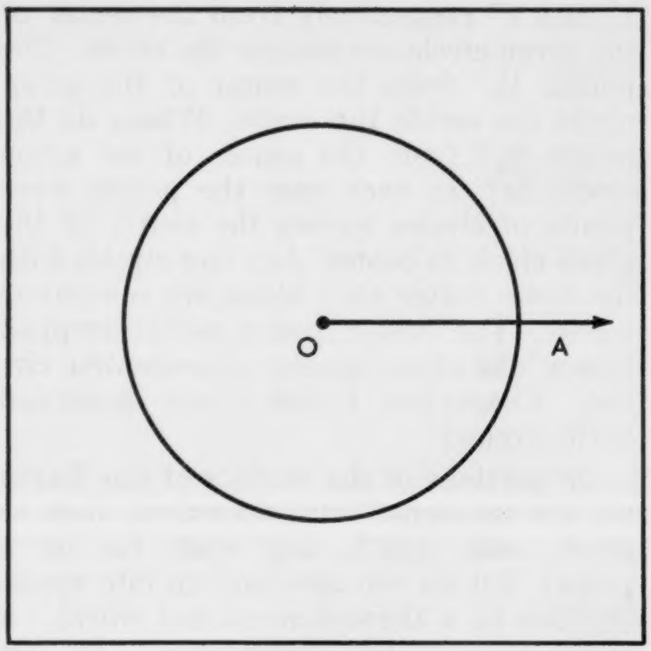


Figure 1–7

Try an experiment. On a piece of cardboard draw a circle with center O and a ray OA as in Figure 1–7. Next cut out a pointer as shown by OB in Figure 1–8 and use a pin to attach one end of the pointer so that it can rotate freely about from the point O.

Several possible positions of the pointer are shown in Figure 1–9. Think of the

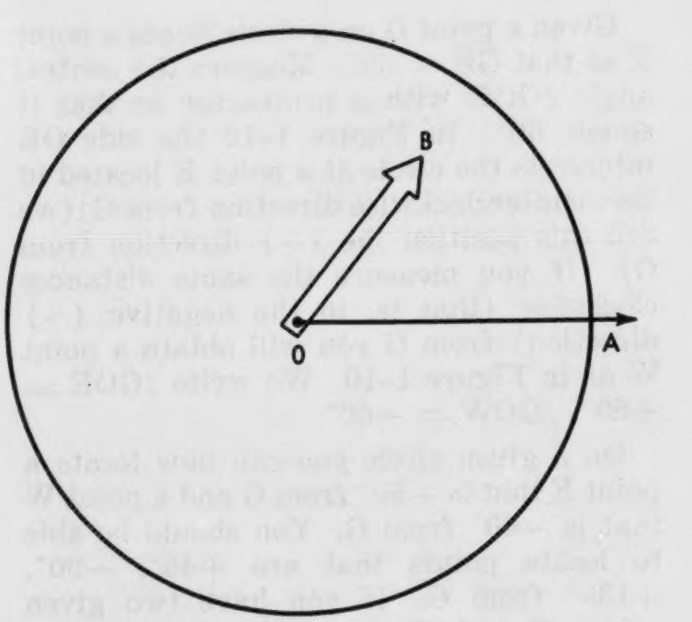


Figure 1–8

pointer as starting in position OA, rotating counterclockwise through positions OB, OC, OD, OE, OF and OG where OG is on OA; that is, making one complete rotation.

There are angles and arcs associated with each part of Figure 1–9. One convenient unit of measure is one *revolution* which is the measure of one complete rotation. The most common unit of measure is the *degree*—

360 degrees = 1 complete revolution.

Estimate the measures of the angles in Figure 1–9 in revolutions and in degrees. Your answers should be approximately the following:

∠AOA = 0 revolutions = 0°
∠AOB = $\frac{1}{12}$ revolution = 30°
∠AOC = $\frac{1}{4}$ revolution = 90°

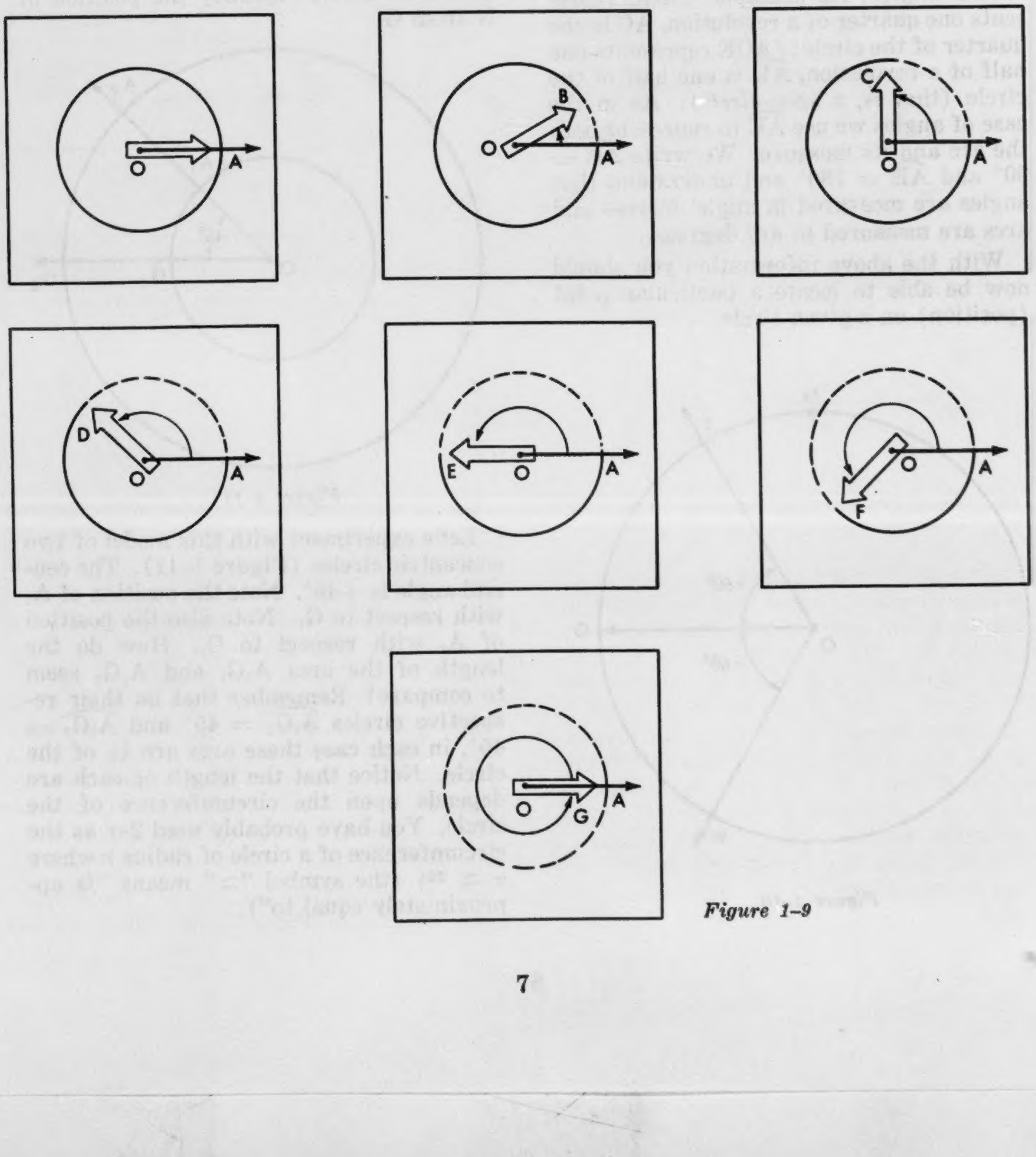


Figure 1–9

$\angle AOD = \frac{3}{8}$ revolution $= 135°$
$\angle AOE = \frac{1}{2}$ revolution $= 180°$
$\angle AOF = \frac{5}{8}$ revolution $= 225°$
$\angle AOG = 1$ revolution $= 360°$

Notice in the above list that the symbol for an angle, such as $\angle AOB$ is also used to describe the measure of the angle, ($\angle AOB = 30°$). This ambiguity is sometimes avoided by using the symbol "m $\angle AOB$" for the measure of the angle but we shall not do so.

Notice also that the rays $\overrightarrow{OA}$ and $\overrightarrow{OC}$ form an angle of 90° and are said to be *perpendicular;* we write $\overrightarrow{OA} \perp \overrightarrow{OC}$ (read "OA is perpendicular to OC").

The measures of arcs $\overset{\frown}{AC}$, $\overset{\frown}{AD}$, $\overset{\frown}{AE}$, $\overset{\frown}{AF}$ and $\overset{\frown}{AG}$ in Figure 1-9 may be expressed in terms of the measures of their *respective central angles;* for example: $\angle AOC$ represents one quarter of a revolution, $\overset{\frown}{AC}$ is one quarter of the circle; $\angle AOE$ represents one half of a revolution, $\overset{\frown}{AE}$ is one half of the circle (that is, a *semi-circle*). As in the case of angles we use $\overset{\frown}{AE}$ to represent both the arc and its measure. We write $\overset{\frown}{AC} = 90°$ and $\overset{\frown}{AE} = 180°$ and understand that angles are measured in angle degrees and arcs are measured in arc degrees.

With the above information you should now be able to locate a particular point (position) on a given circle.

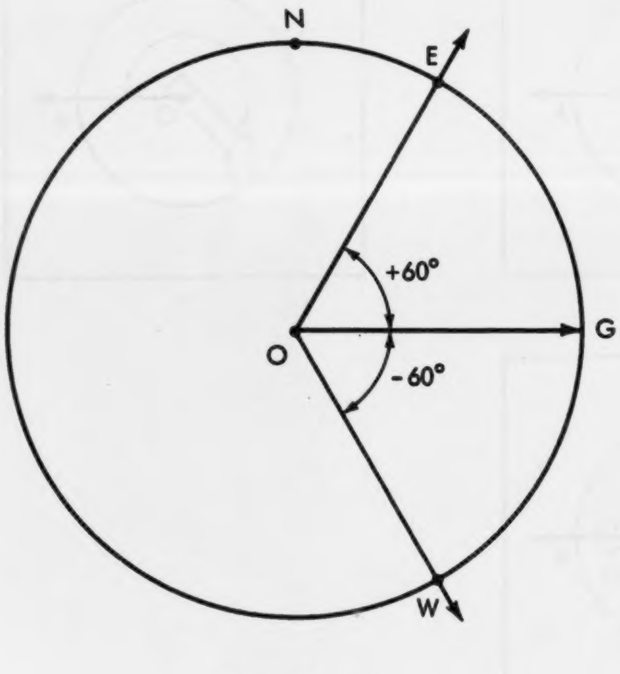


Figure 1–10

Given a point G on a circle locate a point E so that $\overset{\frown}{GE} = 60°$. Measure the central angle $\angle GOE$ with a protractor so that it equals 60°. In Figure 1–10 the side OE intersects the circle at a point E located in the counterclockwise direction from G; (we call this position the (+) direction from G). If you measure the same distances clockwise (that is, in the negative (−) direction) from G you will obtain a point W as in Figure 1–10. We write $\angle GOE = +60°$, $\angle GOW = -60°$.

On a given circle you can now locate a point E that is +60° from G and a point W that is −60° from G. You should be able to locate points that are +45°, −90°, +135° from G. If you have two given points G and N (any other point), you should be able to identify the position of N from G.

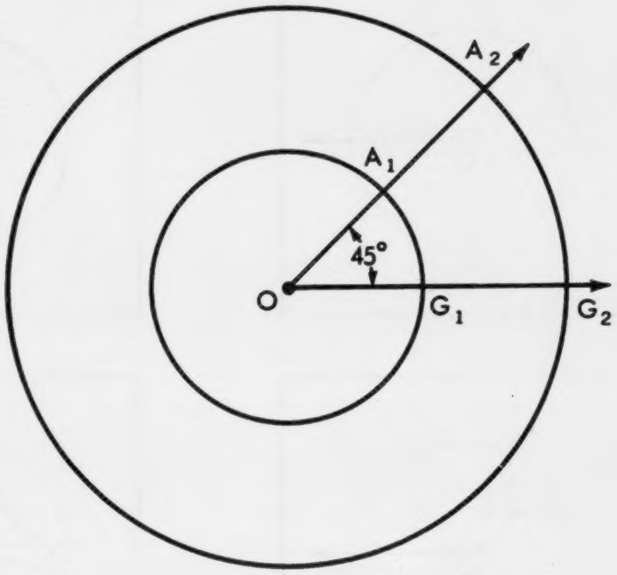


Figure 1–11

Let's experiment with this model of two concentric circles (Figure 1–11). The central angle is +45°. Note the position of A_1 with respect to G_1. Note also the position of A_2 with respect to G_2. How do the length of the arcs A_1G_1 and A_2G_2 seem to compare? Remember that on their respective circles $\overset{\frown}{A_1G_1} = 45°$ and $\overset{\frown}{A_2G_2} = 45°$, in each case these arcs are $\frac{1}{8}$ of the circle. Notice that the length of each arc depends upon the circumference of the circle. You have probably used $2\pi r$ as the circumference of a circle of radius r where $\pi \approx 22/7$ (the symbol "$\approx$" means "is approximately equal to").

Does $\widehat{A_1G_1}$ in Figure 1–11 appear to equal the radius of its circle? $\widehat{A_2G_2}$? The answer is no to both of these questions for actually each arc is slightly shorter than the radius.

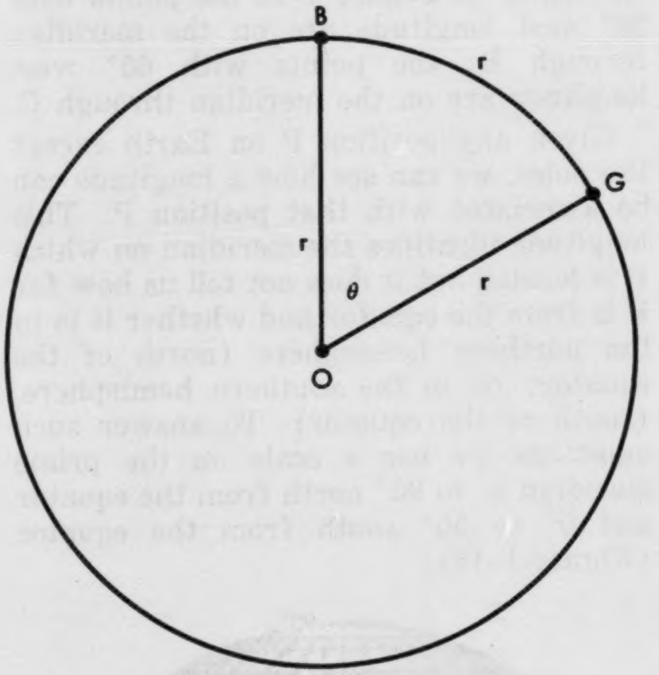


Figure 1–12

In Figure 1–12 notice that $\widehat{GB}$ has length equal to r. We may use the arc GB as a unit of measure for the circumference. Do you see that the circumference then has measure 2π relative to $\widehat{GB}$ as a unit? The central angle of $\widehat{GB}$ (marked θ) is often used as a unit of measure of angles; it is called a radian. Any central angle of 1 *radian* has an arc with the same length as the radius of the circle;

$1 \text{ radian} = 180/\pi \text{ degrees} \approx 57.27°$.

$2\pi \text{ radians} = 1 \text{ revolution} = 360 \text{ degrees}$

In a circle of radius r a central angle of 1 radian has an arc of length r; a central angle of 2 radians has an arc of length 2r; and so forth. We summarize such statements in the formula $d = r\theta$

1–3 Exercises

Angles and Arcs

1. The radius of Earth is about 4,000 miles. What is the approximate circumference of the equator in miles?
2. How long is one radian of arc of the equator in miles?
3. How long is one degree of arc of the equator in miles?

1–4 Positions on Earth

We will assume that Earth is approximately spherical in shape. Remember that a *sphere* is the set of all points in space at a fixed distance from a given point. The radius of Earth is about 4,000 miles. The maps that you probably call World globes illustrate the spherical shape of Earth. In order to locate your town, or other towns, on such a globe, you must visualize positions on the globe in relation to positions on Earth.

Describe the shortest path from one position to another on the surface of the sphere? How are these paths measured? How could we distinguish one town's location as being different from that of any other town?

Let's try an experiment to help us visualize "paths" on the surface of a sphere. Take a spherical ball, or a slate globe; pick any two points A and B that are not on a line through the center of the sphere; draw several paths between these two points. Can you decide which path is the shortest? There are no *straight-line* paths on the surface of a sphere. Each

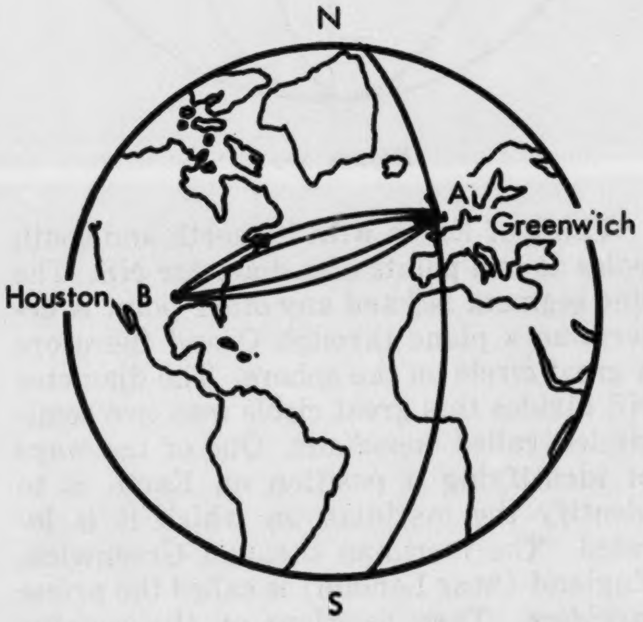


Figure 1–13

path that is on a sphere is an *arc* of a circle just as any slice of orange has a circular shape.

The two points A and B and the center O of the sphere were assumed to be not on a line. Therefore, they determine a plane. This plane AOB intersects the sphere in a circle. Slice through an orange or apple any way you like and see if you do not get a circular slice. However, this circle has the center of the sphere as its center and the radius of the sphere as its radius. Since there are no larger circles on the sphere, it is called a *great circle.* Every plane through the center of a sphere intersects the sphere in a **great circle.** *The shortest path between two points on a sphere is along an arc of a great circle.*

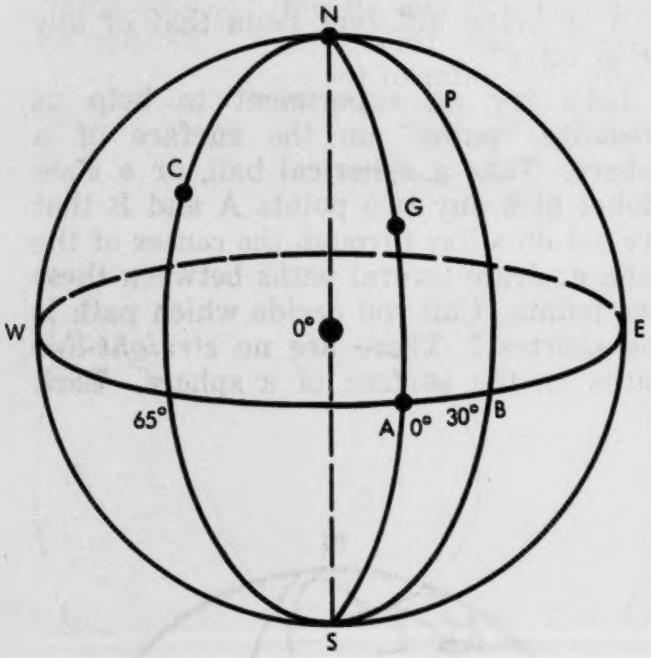


Figure 1–14

Think of Earth with its north and south poles as end points of a diameter NS. The line segment NS and any other point A determine a plane through O and therefore a great circle on the sphere. The diameter NS divides this great circle into two semicircles, called *meridians.* One of the ways of identifying a position on Earth is to identify the meridian on which it is located. The meridian through Greenwich, England (near London) is called the *prime meridian.* Then positions on the equator are located as in Section 1–3 with the intersection of the equator and the prime meridian as the reference point, 0°. Each meridian is identified in terms of a number of degrees (measured along the equator) east or west of the prime meridian; this number is the *longitude* of all points on the meridian. In Figure 1–14 the points with 30° east longitude are on the meridian through B; the points with 65° west longitude are on the meridian through C.

Given any position P on Earth except the poles, we can see how a longitude can be associated with that position P. This longitude identifies the meridian on which P is located but it does not tell us how far P is from the equator and whether it is in the northern hemisphere (north of the equator) or in the southern hemisphere. (south of the equator) To answer such questions we use a scale on the prime meridian 0° to 90° north from the equator and 0° to 90° south from the equator (Figure 1–15).

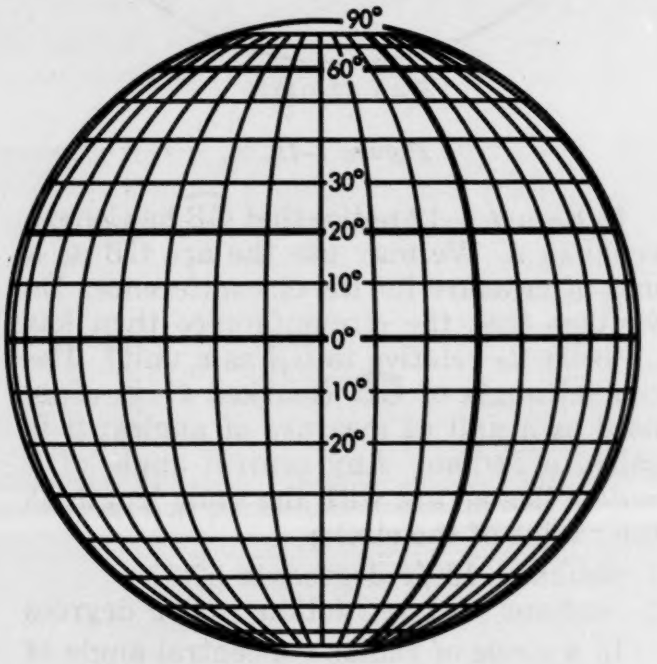


Figure 1–15

If we used scales on all meridians, we would find that all points 20° north of the equator are on a circle (not a great circle) that intersects the prime meridian at 20°; all points 60° north of the equator are on a circle that intersects the prime meridian at 60°, etc. The numbers that indicate degrees north or south of the equator are *latitudes.* Each circle of points with a given latitude is on a plane that is parallel to (does not intersect) the plane of the equator.

You should now be able to visualize that the north pole is at 90° north latitude; the south pole is at 90° south latitude; each point of the equator has 0° latitude and its position can be described by its longitude. Each point on Earth has a position that can be described by its longitude and its latitude.

1–4 Exercises
Positions on Earth

1. Our Earth makes a complete turn of 360° in 24 hours. Through how many degrees does Earth turn in 1 hour? In 6 hours? In 12 hours?
2. The longitude of the prime meridian is 0°. What is the longitude in degrees of the meridian halfway around the world from the prime meridian?
3. The north-south distance around Earth is about 24,860 miles. (a) What is the approximate distance from the north pole to the south pole? (b) From each pole to the equator?
4. If two people should travel east or west around Earth each one remaining on a line of a different latitude, would their routes always be the same distance apart?
5. If two people should travel north or south from the equator on lines of two different longitudes, would their routes always be the same distance apart?
6. This array shows the approximate length in miles of 1° of longitude at different latitudes:

Latitude	Miles in 1° of Longitude
0°	69
10°	68
20°	65
30°	60
40°	53
50°	45
60°	35
70°	24
80°	12
90°	0

How far would a man travel if he went around the world at a latitude (a) of 30° (b) of 50°?

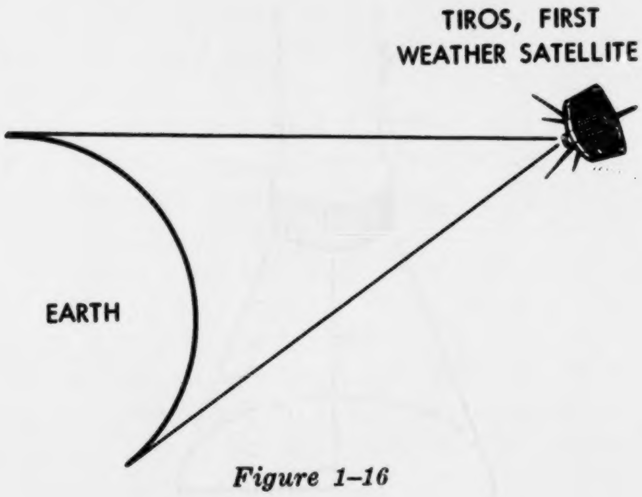


Figure 1–16

1–5 Observations of Earth

If you were asked to locate points ¼″ outside a given circle, how would you locate them? You should measure from a point of the circle along a line that contains the center of the circle.

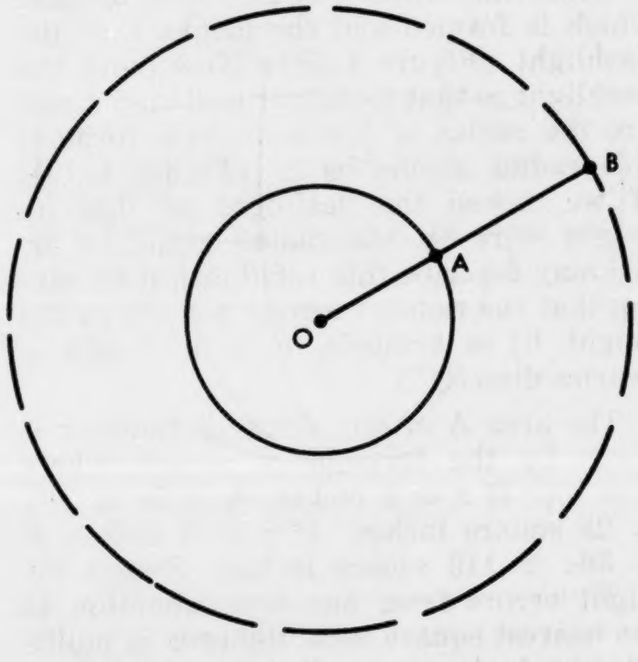


Figure 1–17

In Figure 1–17 the point O is the center of Earth; A is a point on the surface of Earth, and B is a satellite, or a star, planet, or object in the sky. The length AB is the height or *altitude* of the satellite above Earth's surface. How do you think we could find how much of Earth's surface can be seen from a satellite at a given altitude?

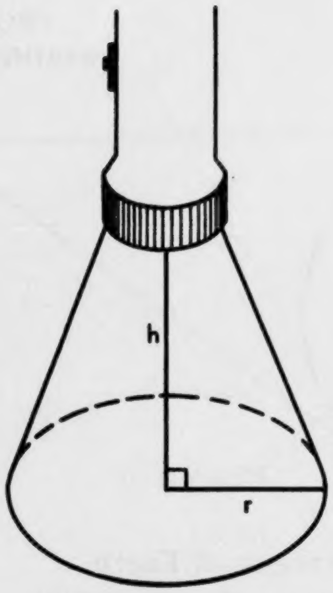


Figure 1–18

Let's try an experiment. Take a sheet of paper and from some point above shine a flash light straight down on the paper. Measure the radius r of the circle of light which is formed and the height h of the flashlight (Figure 1–18). Now move the flashlight so that its height is 2h and measure the radius of the new circle formed; this radius should be 2r (Figure 1–19). If we moved the flashlight so that its height were 3h, the radius would be 3r. We may describe this relationship by saying that the radius r *varies directly as* the height h; in symbols, $r \propto h$ (read $\propto$ "varies directly").

The area A of any circle of radius r is given by the formula $A = \pi r^2$, where $\pi \approx 22/7$. If $r = 3$ inches, $A = 9\pi \approx 198/7 \approx 28$ square inches. If $r = 6$ inches, $A = 36\pi \approx 113$ square inches. Except for slight errors from our approximation to the nearest square inch, the area is multiplied by 4 when the radius is multiplied by 2; similarly, the area is multiplied by 9 when the radius is multiplied by 3. In general, the area varies directly as the square of the radius, $A \propto r^2$. We could have anticipated this from the equation $A = \pi r^2$.

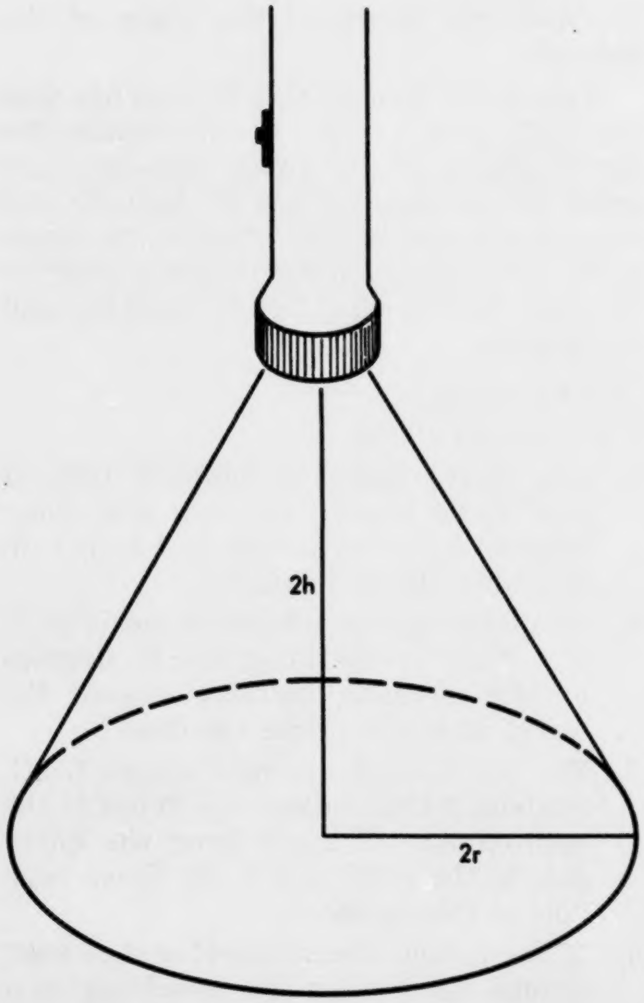


Figure 1–19

When we observe Earth from a satellite we are essentially observing a part of the surface of a sphere. The formula for the surface S of a sphere of radius r is $S = 4\pi r^2$ as you have probably already studied. Since the radius of Earth is about 4,000 miles, the surface of Earth is about 201,000,000 square miles. An understanding of the vastness of Earth's surface should help us understand some of the problems of weathermen. Too often the weatherman has been blamed for his poor weather predictions but regular weather observations cover only about one-fifth of Earth's surface and the forecaster may be unaware of changing conditions on other parts of Earth's surface which might alter his predictions.

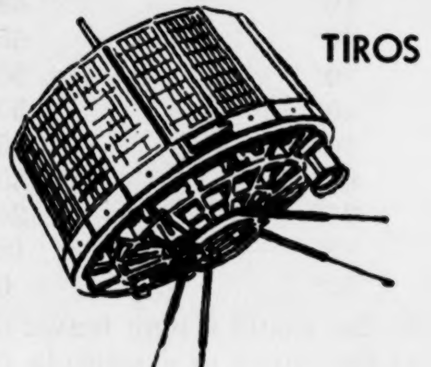


Figure 1–20

TIROS (a NASA weather satellite) can circle the Earth in little over an hour and transmit images of cloud formations. Forecasters analyze and interpret this information and can give more accurate weather predictions. Other valuable services have been performed by this worldwide observation. Once a hoard of locusts approaching a section of Africa were sighted in time to warn the farmers who gathered their crops and had them safely stored away before the locusts arrived.

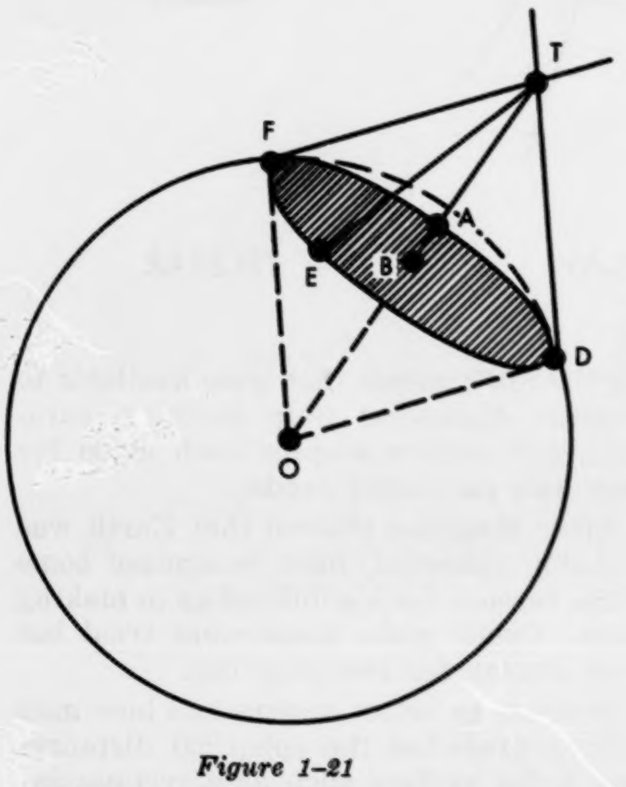
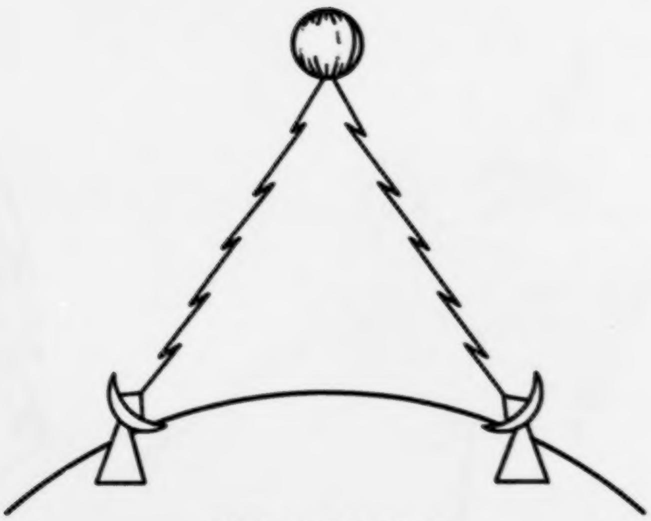


Figure 1–21

The Earth is observed from a satellite such as Tiros. However it can "see" only a part of Earth in any one observation. The shaded part in Figure 1–21 represents the part of Earth's surface that can be seen when Tiros is at a position T. We think of T as directly above a point A on Earth; that is, on a ray OA where O is the center of Earth. From T the points D, E, and F appear to be on the edge (horizon) of the Earth. Actually $\overline{TD}$ is perpendicular to $\overline{OD}$. Then since $\overline{TD}$ has only the point D on the sphere representing Earth, the line TD is said to be *tangent* to the sphere; $\overline{TE}$ and $\overline{TF}$ are also tangent to the sphere. The points D, E, and F are on a circle (not a great circle) with center B. As in the case of circles of constant latitude (Section 1–4), the size of this circle depends upon the distance OB. Since $OB + AB = OA \approx 4{,}000$ miles, the radius of Earth, the size of the circle also depends on AB. The distance AB is called the height h of the *zone* (part of the surface of the sphere) bounded by the circle. Think of cutting off a part of an orange; the deeper the cut, the more of the surface you cut off. The formula for the area of zone will be used in Chapter 3 and is $A = 2\pi rh$. Thus when a satellite is high enough to observe a zone of height one mile, over 25,000 square miles of Earth's surface can be observed ($2 \times \frac{22}{7} \times 4{,}000 \times 1 > 25{,}000$).

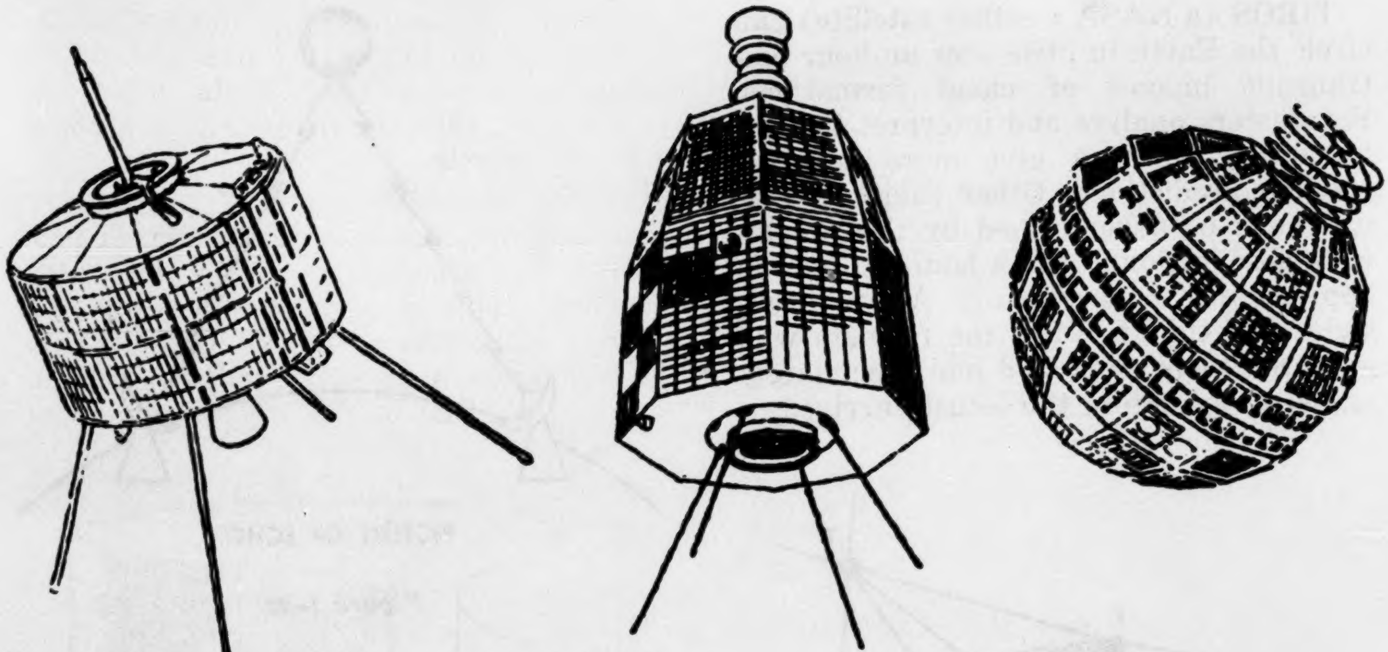


PICTURE OF ECHO

Figure 1–22

The points of a zone on Earth's surface can be observed from a satellite as in Figures 1–16 and 1–21. Notice also that the satellite can be observed from points of this same zone. In 1960 NASA demonstrated that radio signals could be reflected off the man-made satellite, Echo I, and received several thousand miles away. Echo I was used to "bounce" two-way voice conversations and other communication data of good quality across the United States, and between this country and Europe. Echo is referred to as a *passive satellite* for it simply reflects or "bounces" a message from one point on Earth to another.

Satellites such as Relay, Syncom and Telstar are called *active* or "repeater"



SYNCOM RELAY TELSTAR

Figure 1–23

satellites because they receive, amplify, and rebroadcast messages transmitted to them.

1–5 Exercises

Observations of Earth

1. Find the area of the northern hemisphere if the radius of Earth is approximately 4000 miles.
2. If a satellite's cameras can "see" about one-fifth of the surface of Earth during each day, how many square miles can be photographed each day?
3. Use 4,000 miles as the radius of Earth and $A = 2\pi rh$ as the formula for the area of a zone to find the area of Earth that can be photographed with a camera that can "see" a zone with height 20 miles.

1–6 Maps and Distances

Man has made maps throughout recorded history. Early map makers made maps of a very small part of Earth's surface; each map maker's location was generally the center of the map. Many early maps were surprisingly accurate considering the instruments that were available to measure distances. Ship captains, caravans, and armies adopted such maps for their own particular needs.

After Magellan showed that Earth was probably spherical, man recognized some of the reasons for his difficulties in making maps. Crude globe maps were tried but were clumsy for everyday use.

In order to better understand how map makers projected the spherical distances onto a flat surface such as a rectangular sheet of paper, try this experiment. You will need two meter sticks, two pieces of string, and two small weights. Tie the weighted ropes 50 centimeters apart on one of the meter sticks. Place the other meter stick on the edge of a table. Then hold the first meter stick parallel to the floor so that the distance between the weighted strings can be measured on the meter stick on the table. Notice that this distance is also 50 centimeters. (Figure 1–24)

Tilt the first meter stick so that it makes an angle of about 30° with the floor and note the distance on the second stick. Tilt the first meter stick at a 45° angle and note the distance. Try several other angles

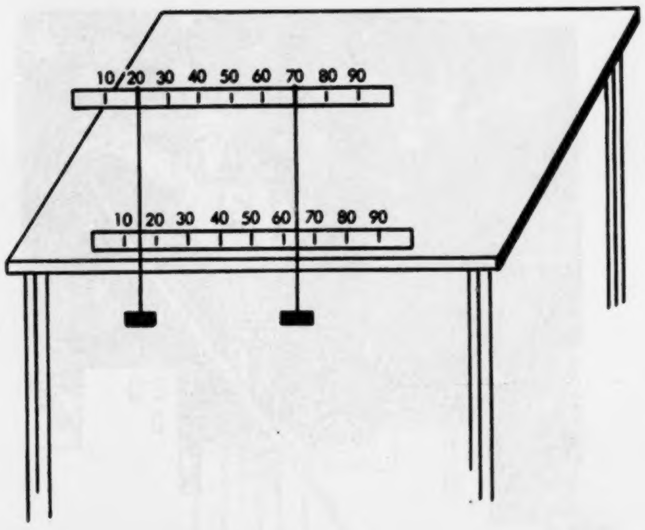


Figure 1–24

of inclination. Notice that the distance decreases as the angle increases from 0° to 90°.

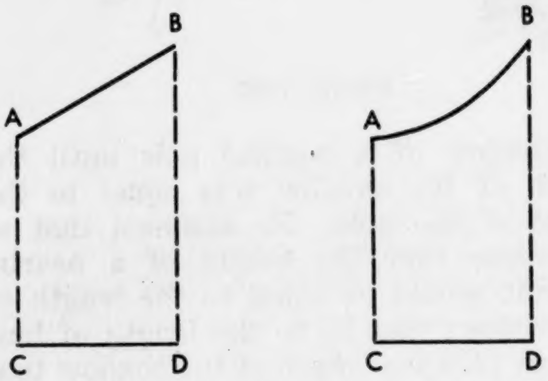


Figure 1–25

Now think of yourself as a map maker (a cartographer). You can project a line segment AB or an arc AB onto a line segment CD. You use parallel lines AC and BD perpendicular to (that is, at right angles to) the line or plane on which $\overline{CD}$ is to be drawn to find the points C and D just as the weighted strings were used in the experiment. Since these construction lines are perpendicular to the line or surface on which the drawing is made, we speak of the mapping of $\overline{AB}$ onto $\overline{CD}$ as an *orthogonal projection.* Notice that projections can distort the lengths of the line segments.

Consider orthogonal projections of a circle. If the plane of the circle is parallel to the plane onto which you are projecting, the projection is also a circle; if the planes are perpendicular, the projection is a line segment congruent to a diameter of the circle; if the planes are inclined at an angle which we will call "theta" θ and ($0° < \theta < 90°$), then the projection is a curve but not a circle.

There are other types of projections. For example, the image on a film is projected onto a screen on which you watch a movie. This is essentially a projection from a point (the source of light). A projection from a point is called a *central projection.* One way to obtain a map of

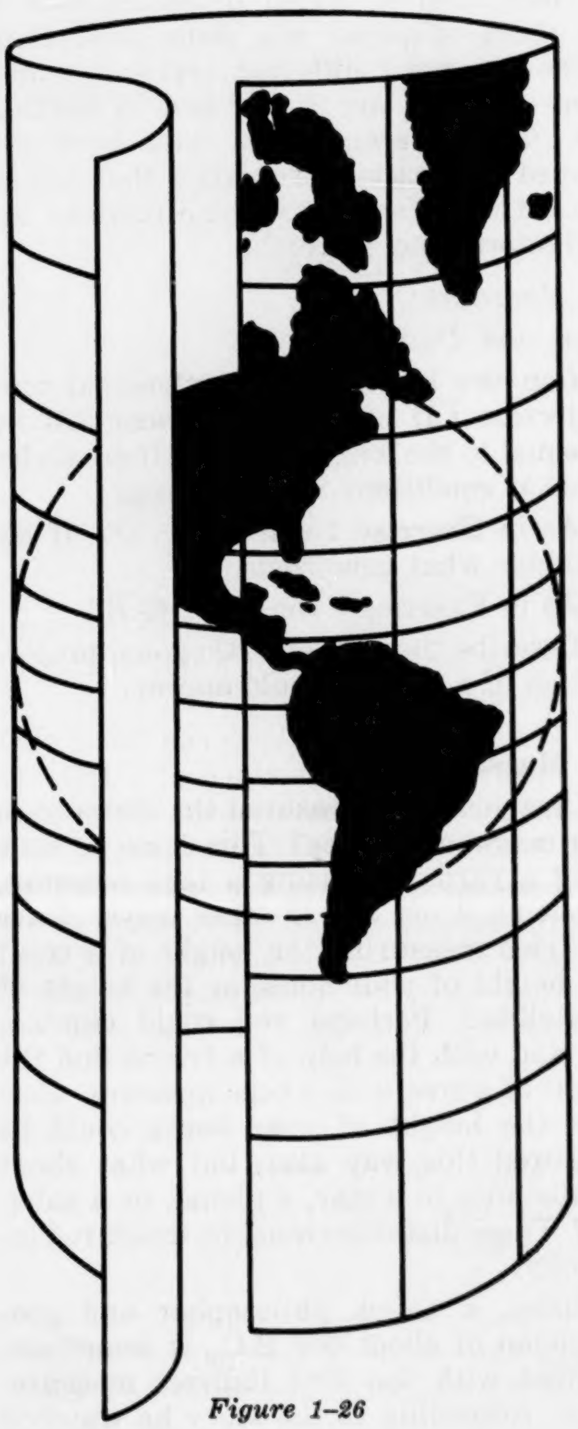


Figure 1–26

part of the surface of the Earth on a flat sheet of paper is to think of a projection from the center of Earth onto the paper rolled in a cylinder around Earth (Figure 1–26); then cut the cylinder and place the paper flat. When an image on the film for a movie is projected on a screen, the image on the screen is much larger than the one on the film; size has been changed (distorted) so you can see the image better. When a map of Earth is obtained by central projection both size and shape are distorted.

Every map of Earth projected onto a flat sheet of paper has some distortion. There are many different types of maps. Some of these are considered in Section 2–3. The different types have been developed to preserve properties that are of interest to different people; directions for navigators, and so forth.

1–6 Exercises

Maps and Distances

1. Can the length of an orthogonal projection $\overline{CD}$ of a line segment AB be equal to the length of $\overline{AB}$? If so, under what conditions will this occur?
2. As in Exercise 1 can $\overline{CD} = O$? If so, under what conditions.
3. As in Exercise 1 can $\overline{CD} > \overline{AB}$?
4. Describe the way an orthogonal projection of a sphere would appear.

1–7 Measurements

Have you ever measured the distance to your neighbor's house? This could be done using a yardstick, using a tape measure, by pacing it off, and in other ways. Have you tried measuring the height of a tree? The height of your house or the height of a satellite? Perhaps you could climb a tree and with the help of a friend find the height of a tree with a tape measure. Perhaps the height of your house could be measured this way also; but what about the distance to a star, a planet, or a satellite? These distances must be measured indirectly.

Thales, a Greek philosopher and geometrician of about 600 B.C., is sometimes credited with the first indirect measurement. According to the story he watched the shadow of a vertical pole until the length of the shadow was equal to the height of the pole. He assumed that at this same time the height of a nearby pyramid would be equal to the length of its shadow; that is, to the length of half its base plus the length of the shadow that extended beyond the base (Figure 1–28). He could measure both of these distances and thus he could find the height of the pyramid.

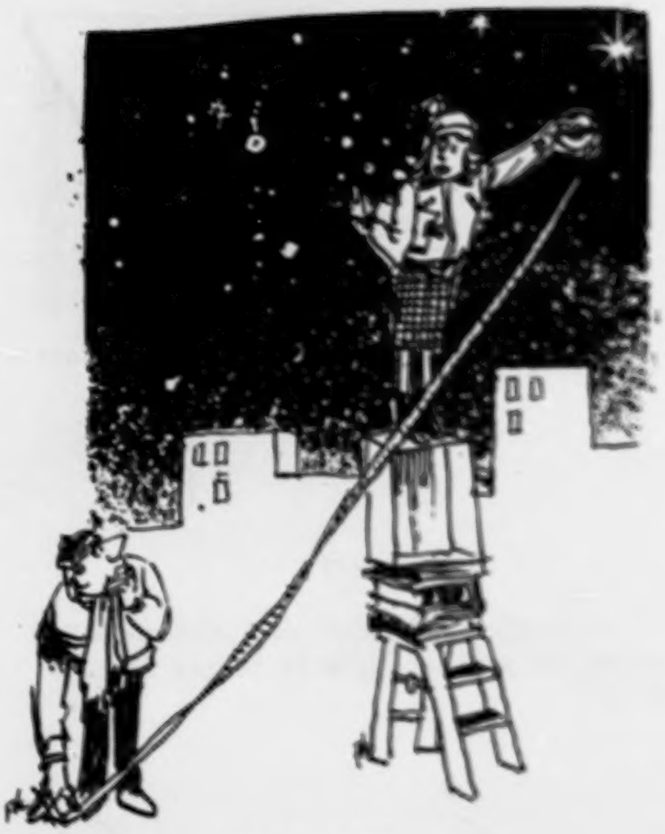


Figure 1–27

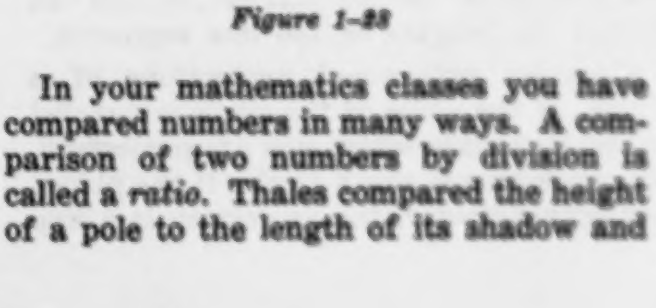


Figure 1–28

In your mathematics classes you have compared numbers in many ways. A comparison of two numbers by division is called a *ratio*. Thales compared the height of a pole to the length of its shadow and

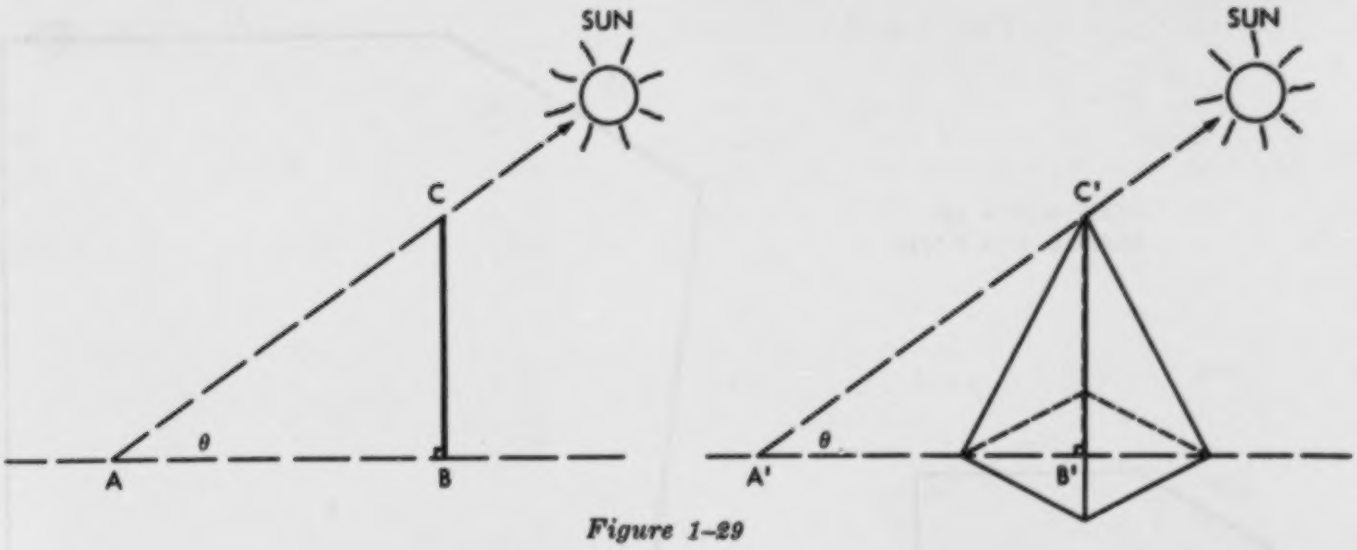


Figure 1–29

the height of a nearly square pyramid to the length of its shadow. He assumed that these measures were equal and thus in a 1 to 1 ratio. Notice that Thales could have measured the pole and its shadow in inches and measured the height of the pyramid and its shadow in feet and the ratio would still remain constant.

Thales assumed that the sun's rays would make the same angle with Earth's surface (horizontal) when forming each shadow. In other words, he assumed that the *angle of elevation* of the sun (indicated by θ in Figure 1–29) was the same in both cases. We assume that the pole was in a vertical position and the top of the pyramid was directly over the center of its base. Then the triangles indicated by the dashed lines in Figure 1–29 are right triangles (the angle marked $\llcorner$ of each triangle is a right angle). Also since the angles are *congruent* (have the same measure), the triangles are of the same shape. If you have not already done so, you may later study such triangles as *similar triangles;* their corresponding angles are congruent. (We use the symbol $\cong$ to write congruent):

$$\angle A \cong \angle A', \angle B \cong \angle B', \angle C \cong \angle C';$$

the lengths of their corresponding sides are in a constant ratio:

$$\frac{\overline{AB}}{\overline{A'B'}} = \frac{\overline{BC}}{\overline{B'C'}} = \frac{\overline{AC}}{\overline{A'C'}};$$

and each triangle may be visualized as a picture of the other possibly drawn to a different scale.

Figure 1–30 represents two scale drawings of the same irregularly shaped plot of land. The lengths of the sides are given in inches. The given scales enable us to measure one of the drawings to determine the actual dimensions of the plot of land in feet. Although these drawings are different in size they have the same shape and are said to be *similar* ($\sim$). They are similar to each other and also to the boundaries of the plot of land under consideration. If you measure the angles, you should find that the corresponding angles are congruent. Notice that the lengths of the corresponding sides are in the same ratio:

$$\frac{2}{4} = \frac{1\frac{3}{8}}{2\frac{6}{8}} = \frac{1\frac{5}{8}}{3\frac{1}{4}} = \frac{1}{2} = \frac{\frac{5}{8}}{\frac{10}{8}}.$$

Scale models play an important part in our space program. Accurate scale models are more economical to make than full size replicas. Many things can be studied and interpreted from models. Often changes in design are based upon a study of models.

The three triangles in Figure 1–31 are similar triangles. Each triangle has an angle of 30° and an angle of 60°. Measure the third angle in each triangle. You should find that it is 90° in each case. Since $\triangle ABC$ and $\triangle DEF$ are similar, the lengths of the sides are proportional:

$$\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{AC}}{\overline{DF}} = \frac{\overline{BC}}{\overline{EF}};$$

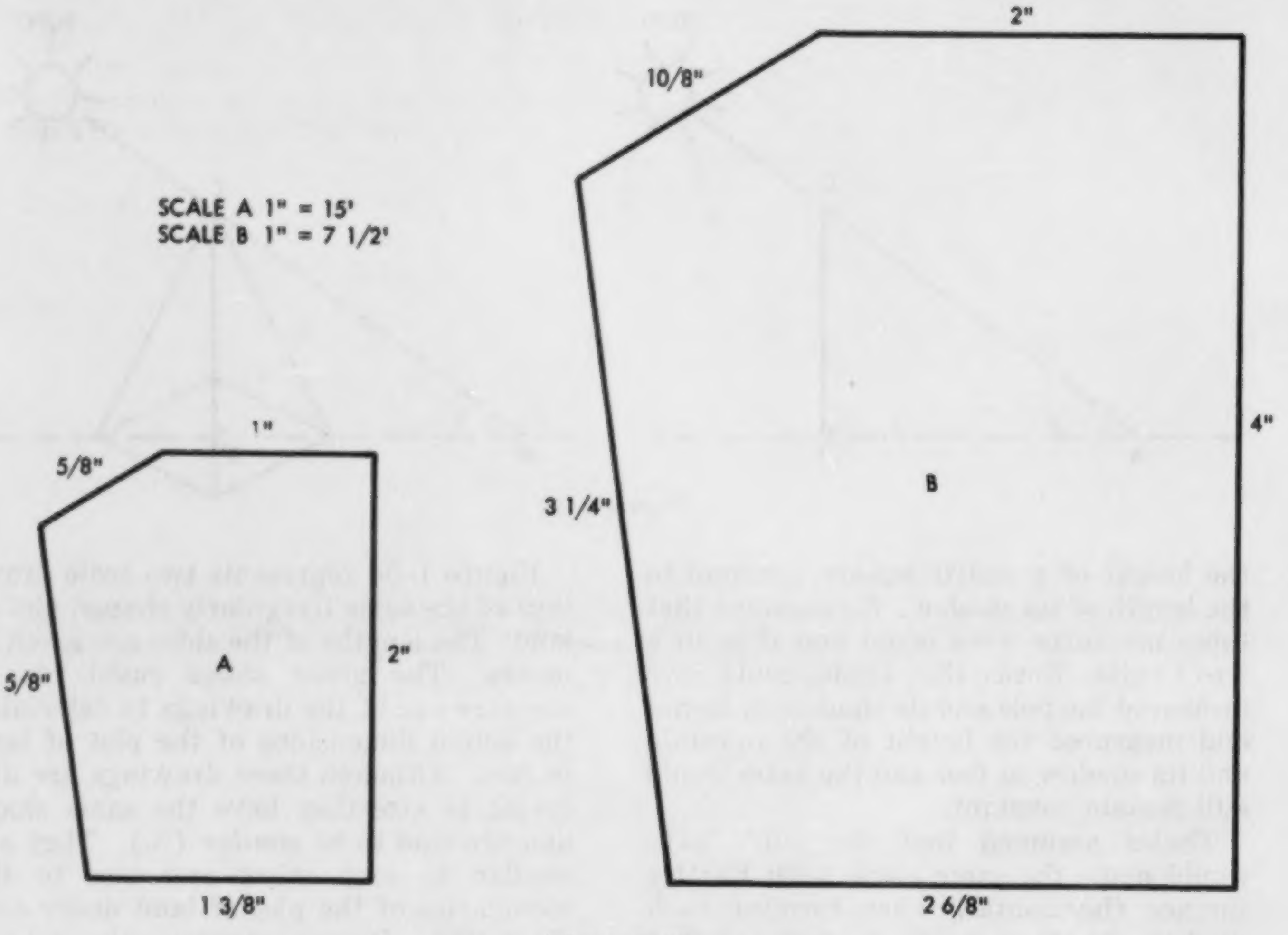

2"
10/8"
SCALE A 1" = 15'
SCALE B 1" = 7 1/2'
4"
B
3 1/4"
2 6/8"
1"
5/8"
A
2"
1 5/8"
1 3/8"

Figure 1–30

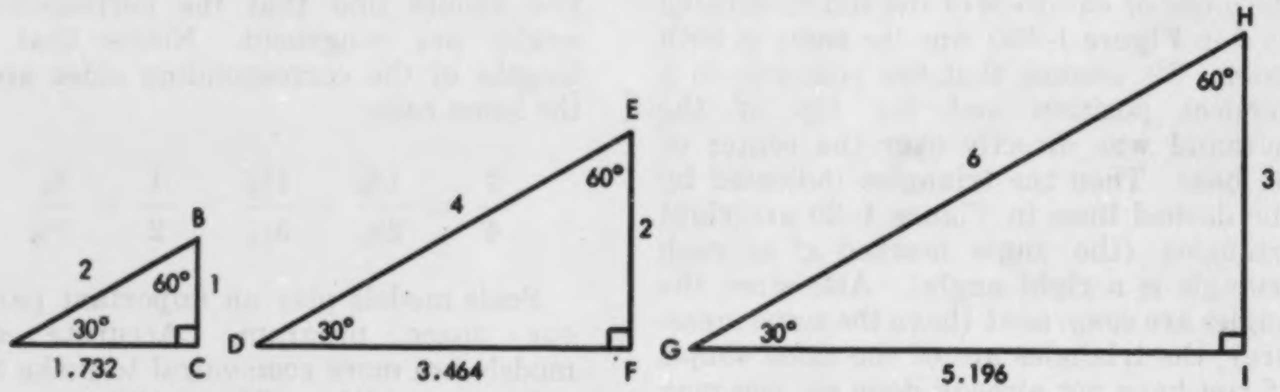

B
2
60°
1
A
30°
C
1.732
E
4
60°
2
D
30°
F
3.464
H
60°
6
3
G
30°
I
5.196

Figure 1–31

since △ABC and △GHI are similar we have

$$\frac{\overline{AB}}{\overline{GH}} = \frac{\overline{AC}}{\overline{GI}} = \frac{\overline{BC}}{\overline{HI}}.$$

In the first case each ratio is equal to ½; in the second case each is equal to ⅓.

Now consider the *proportion* (an equality of two ratios)

$$\frac{\overline{AC}}{\overline{DF}} = \frac{\overline{BC}}{\overline{EF}}.$$

This proportion may also be stated as

$$\frac{\overline{EF}}{\overline{DF}} = \frac{\overline{BC}}{\overline{AC}}.$$

Such ratios of the lengths of two of the sides of a triangle are equal for any two similar triangles and are used so extensively for right triangles that they are given special names.

Think of Figure 1–32 as any right triangle with the right angle at C. The side $\overline{AB}$ opposite the right angle is called the

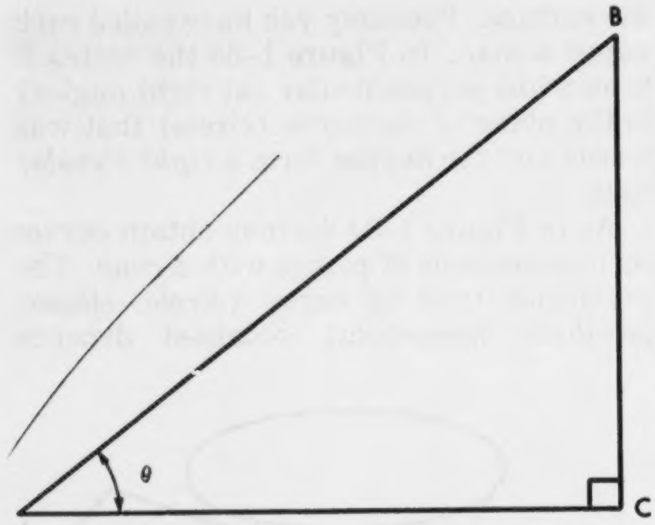


Figure 1–32

hypotenuse of the triangle. The other two sides may be identified either with reference to ∠A or to ∠B. Relative to ∠A we call $\overline{BC}$ the *side opposite* and $\overline{AC}$ the *side adjacent*. We give these ratios special names as shown below. Now if θ is the measure of ∠A, we define the following ratios as:

$$\text{sine} \quad \theta = \frac{\overline{BC}}{\overline{AB}} \qquad \frac{\text{side opposite}}{\text{hypotenuse}}$$

$$\text{cosine} \quad \theta = \frac{\overline{AC}}{\overline{AB}} \qquad \frac{\text{side adjacent}}{\text{hypotenuse}}$$

$$\text{tangent} \ \theta = \frac{\overline{BC}}{\overline{AC}} \qquad \frac{\text{side opposite}}{\text{side adjacent}}$$

These ratios provide the basis for the study of trigonometry and are used extensively in many applications of mathematics. We usually abbreviate

sine θ as $\sin\theta$
cosine θ as $\cos\theta$
tangent θ as $\tan\theta$

Now let us look again at Figure 1–31. Notice that in each triangle:

$$\sin 30^\circ = \frac{1}{2}$$

$$\cos 30^\circ = \frac{1.732\ldots}{2} \approx 0.866$$

$$\tan 30^\circ = \frac{1}{1.732\ldots} \approx 0.577$$

Since for any single angle θ the ratios always have the same values, these values are usually in a table. (See page 184).

1–7 Exercises

Measurements

1. If a vertical 10-foot pole casts a 6-foot shadow, how tall is a tree with an 18-foot shadow?
2. As in Exercise 1 how long a shadow should a person 5-feet tall have?
3. What assumption has been made in Exercises 1 and 2 regarding the positions of the objects and their shadows?

1–8 Paths in Space

Do you believe that "what goes up must come down"? Have you ever shot an arrow up and watched it come down? When you throw a baseball, does its path trace a curve? Does a spinning top trace a curve? What path does the moon take around the earth? What sort of paths do rockets travel? How are the paths of satellites named? Most of these curves are closely related to curves that are studied in geometry. Let's see if we can name some of these curves. First we must define a conical surface.

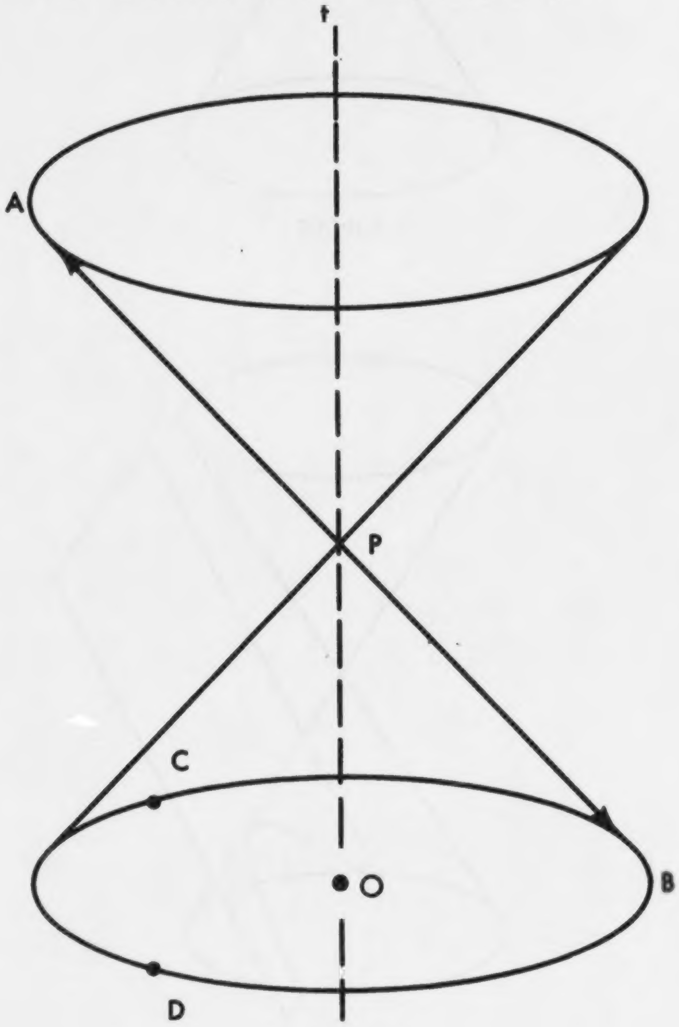
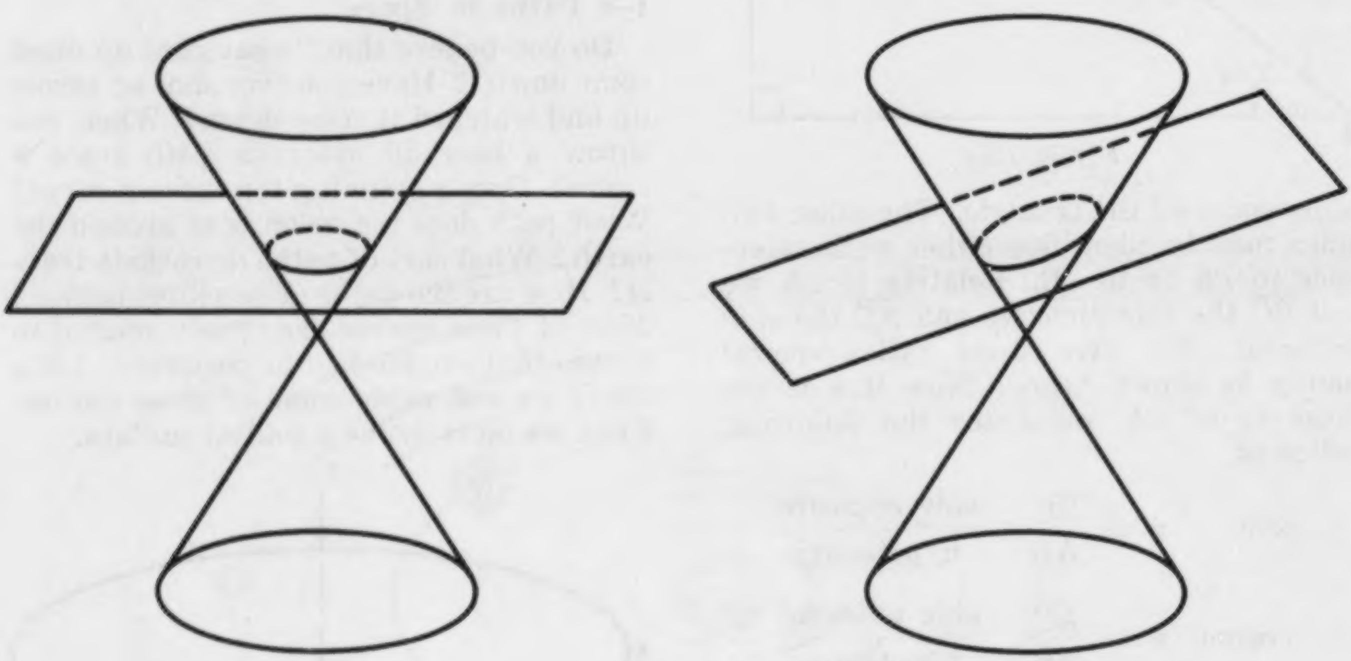


Figure 1–33

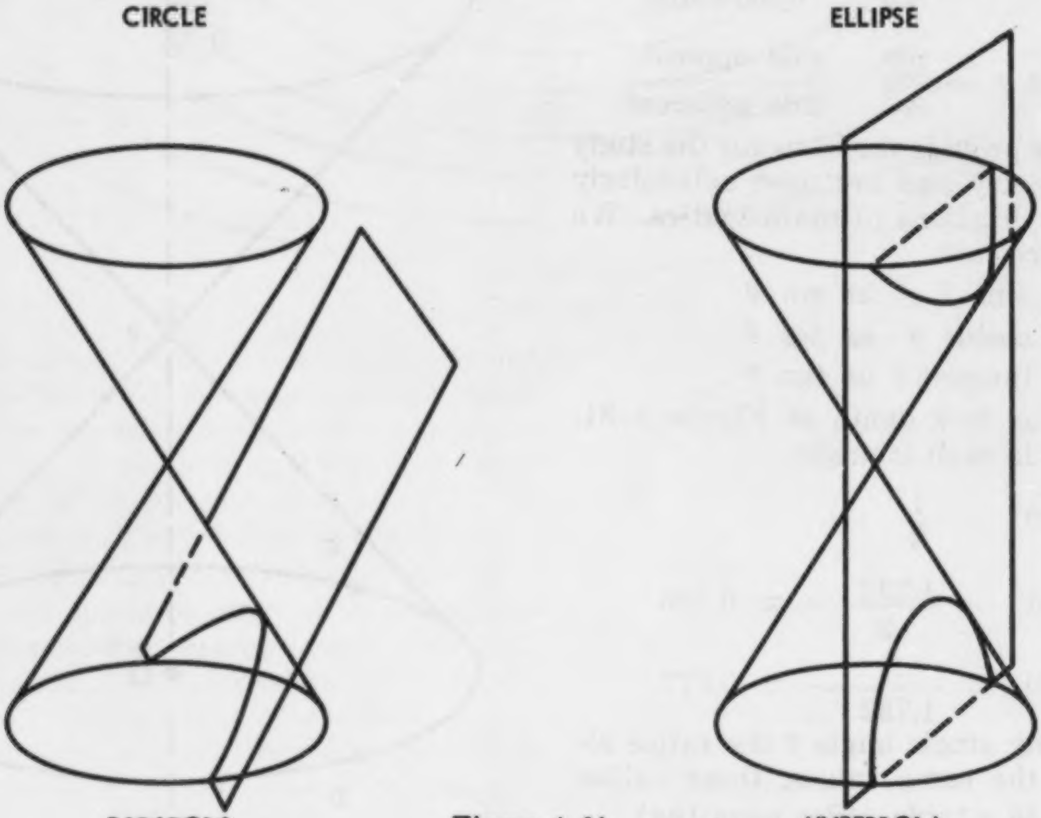
Consider the circle BCD with center O and a line t that is perpendicular at O to the plane of the circle. Let P be any point that is on the line t and different from O. Then think of a line AP that starts in position AB and traces out the circle. In a sense the line is fixed at P and revolves about the circle. The surface generated is called a *conical surface;* the fixed point P is the *vertex* of each of the two *nappes* of the surface. Probably you have called each nappe a *cone.* In Figure 1–33 the vertex P is on a line perpendicular (at right angles) to the plane of the curve (circle) that was traced and the nappes form a *right circular cone.*

As in Figure 1–34 we may obtain curves by intersections of planes with a cone. The particular type of curve (*circle, ellipse, parabola, hyperbola*) obtained depends



CIRCLE

ELLIPSE



PARABOLA

HYPERBOLA

Figure 1–34

upon the angle at which the plane intersects the cone.

The path of sounding rocket is shown in Figure 1–35. If you compare the path of the rocket with the curves in Figure 1–34, you should be able to recognize that it appears to be a parabola as a first approximation.

Can you tell the name of the paths of the planets as they orbit the sun? In Figure 1–36 you should recognize that these paths appear to be ellipses with some of the ellipses almost circles.

In Figure 1–37 you can observe that Earth satellites also have elliptical paths.

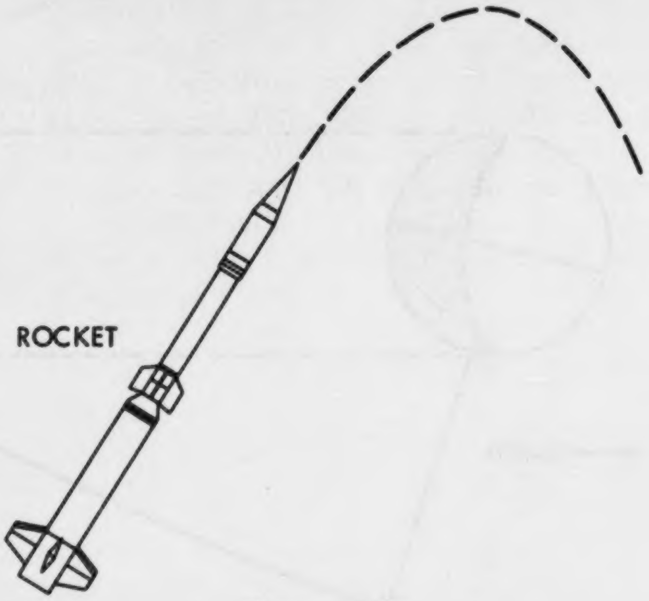


Figure 1–35

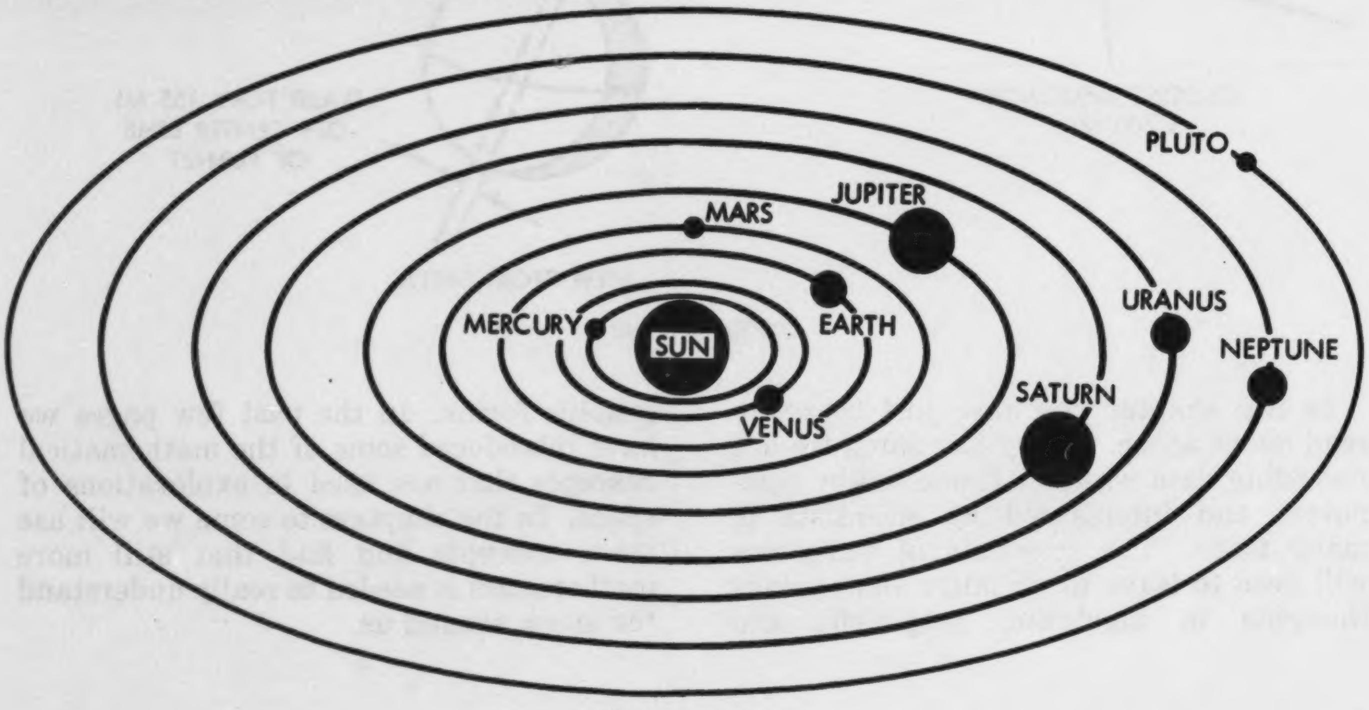


Figure 1–36

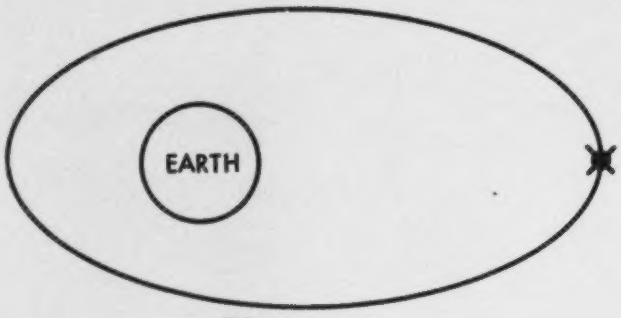


Figure 1–37

Mariner IV was launched November 28, 1964, and put into an orbit about the Sun. In Figure 1–38 Mariner IV was at point A on July 15, 1965, and the cameras were taking pictures at that time. Can you give the name of the curve for the path of Mariner IV? It appears almost straight and we often use a straight line to *approximate* a part of a curve. We could also use a hyperbola to approximate the path of a space probe such as Mariner IV. However, actually the path appears to be elliptical around the sun rather than around Earth.

In Figure 1–38 notice that if we think of this part of the path as along a straight line, then there appears to be similar triangles, ABD and ACE, which we could use in computation to give us additional information. Ellipses, circles, and straight line approximations will be used extensively in later chapters.

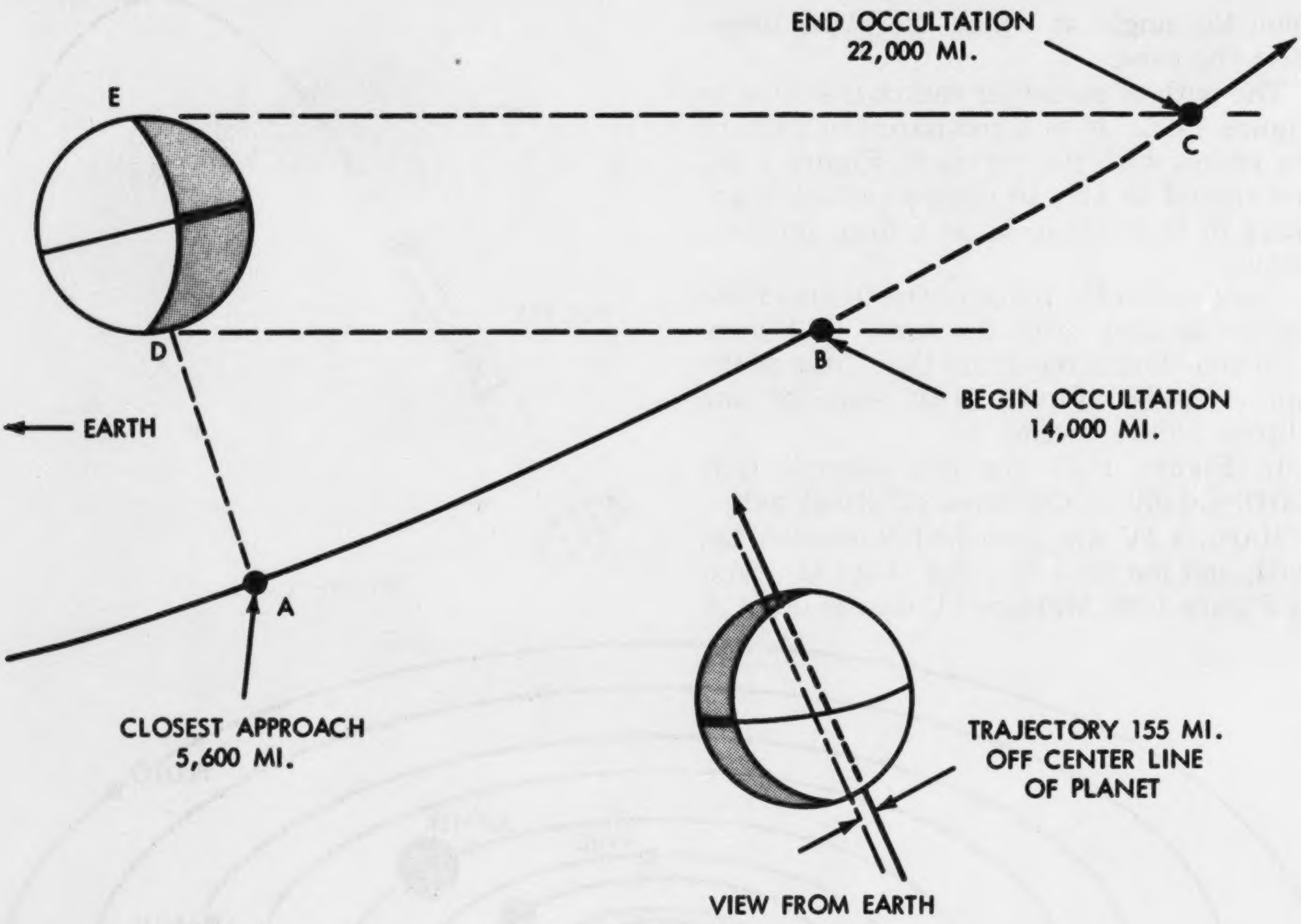
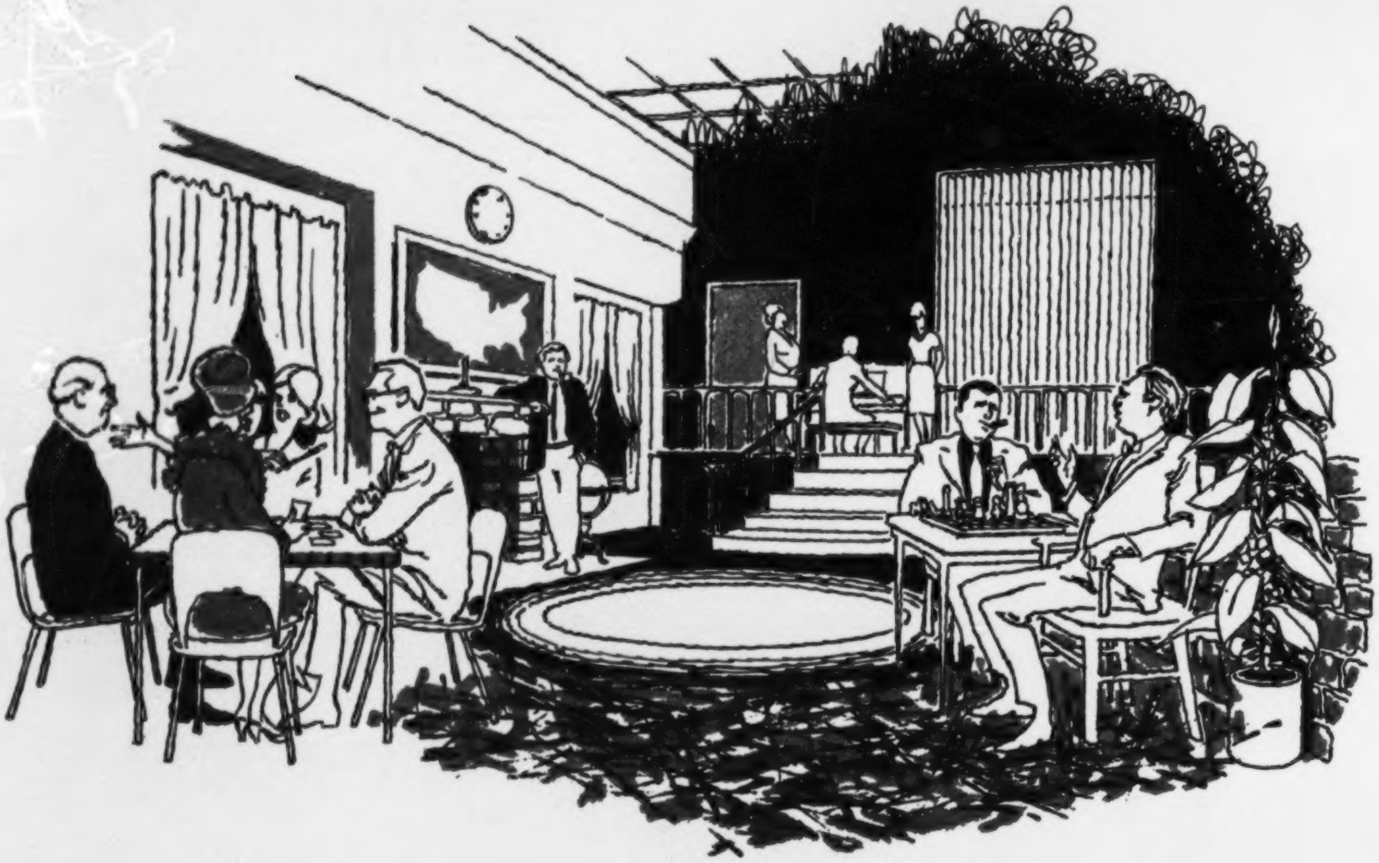

END OCCULTATION
22,000 MI.
E
C
D
B
BEGIN OCCULTATION
14,000 MI.
EARTH
A
CLOSEST APPROACH
5,600 MI.
TRAJECTORY 155 MI.
OFF CENTER LINE
OF PLANET
VIEW FROM EARTH

Figure 1–38

In this chapter you have just begun to read about space. Every day satellites are recording data which is processed by computers and interpreted by scientists in many fields. The scientists of tomorrow will need to learn to visualize and express thoughts in algebraic, geometric, and graphic forms. In the past few pages we have introduced some of the mathematical concepts that are used in explorations of space. In the chapters to come we will use these concepts and find that still more mathematics is needed to really understand the space around us.



Chapter 2

THE UNIVERSE WE LIVE IN

by

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Pittsburgh, Pennsylvania

THE UNIVERSE WE LIVE IN

When we travel on the surface of Earth we can describe our position at any time with only two dimensions. To locate positions in space we need three, four, or perhaps n dimensions. For instance, it is necessary to employ time in describing the position of a satellite as it travels around the Earth. We might think of the cartographers of the future as standing on the shore of the uncharted sea of space, the mapping of which will require new mathematics.

In order to better understand the mapping of space, let's start with something very simple which we already know—a room. Then we will see how far we can proceed by using mathematics to describe the characteristics and conditions of objects in space.

This chapter introduces you to the concept of position as related to some of the coordinate systems used in mathematics. You will need prior knowledge or understanding of coordinate systems. This chapter is intended for readers in the upper elementary grades as well as the high school.

The coordinate systems used are limited to those of ordered pairs:

SYSTEM	ORDERED PAIR
Rectangular	(horizontal, vertical)
Globular Earth	(longitude, latitude)
Polar	(radius vector, vectorial angle)
Navigation	(azimuth, altitude)
Equatorial	(right ascension, declination)

2–1 Where do you live?

The question "where?" is often asked but seldom answered in an exact manner. Consider a small group in a room anywhere on the surface of the earth. Two members are playing chess. Four sit about a card table, and others are gathered around a piano in one corner of the room. A professor is at one side observing what is taking place.

One of the people at the card table has raised a question concerning the meaning of the word "position." Immediately all others, except the professor, expound their theories, each one giving his own interpretation of the word "position." A "status-seeker" might describe position as a place in society, an executive as a place in a company or industrial firm. The "geographer" thinks of position as a location on the earth. One of the chess players—an astronomer, would perhaps describe "position" as a location (point) in space.

Which interpretation do you consider correct? Possibly all are correct. A "mathematics teacher," standing near the piano, while reluctant to enter into the discussion, decided to approach the problem from a mathematical standpoint. All members were asked to find a point in the middle of the wooden door (Figure 2–1) next to the piano. However, the professor questioned the meaning of the "middle" of the door. The question shocked everyone. Surely the professor was joking; for anyone can point to the middle of a door.

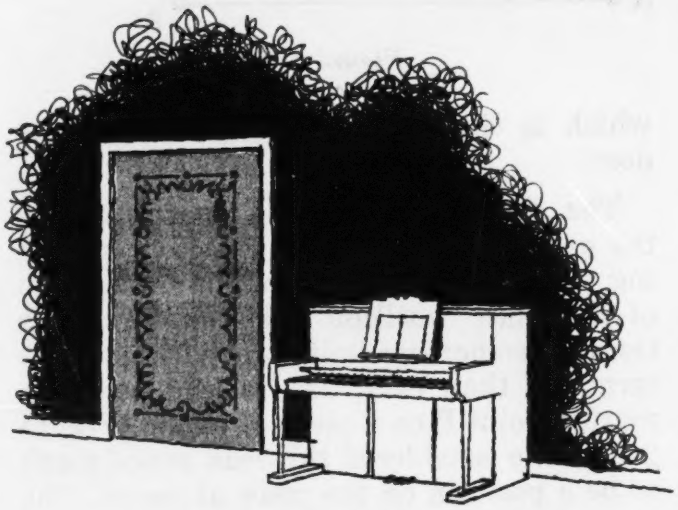


Figure 2–1

However, if you think about this question for awhile, you should realize the significance of the professor's question. It is impossible to look at the middle of a wooden door due to the fact that the door is solid. Therefore the point representing the middle of the door is the intersection of the diagonals of a rectangular solid (Figure 2–2). Since the door is made of wood, a human being cannot see the point

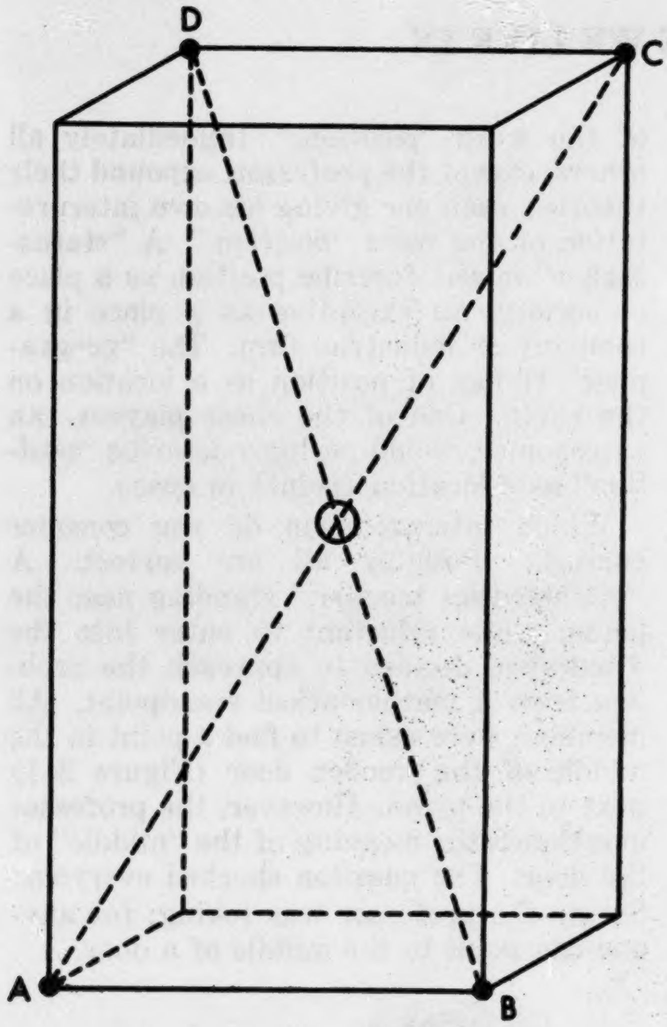


Figure 2-2

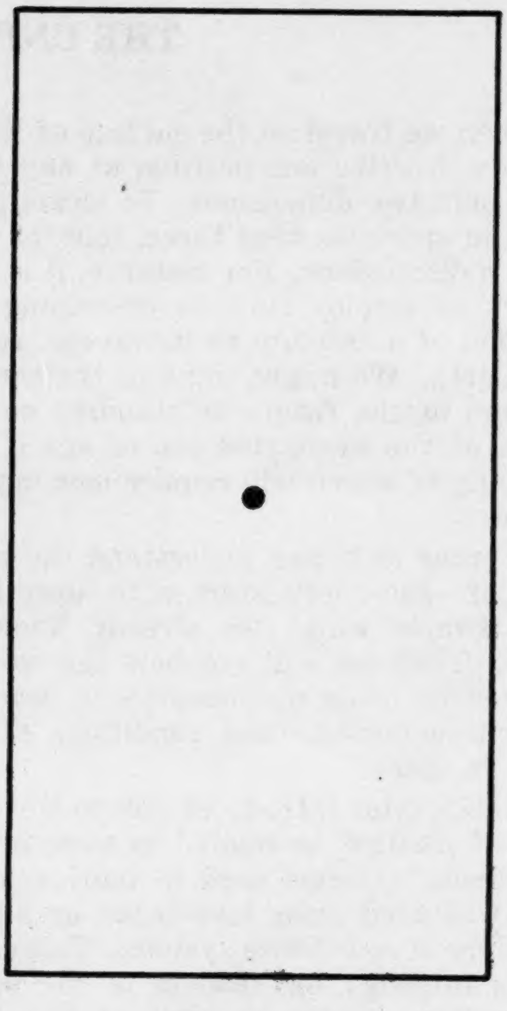


Figure 2-3

which is the middle without cutting the door.

The question about the door provided the group with additional insight concerning the complexities of finding a definition of the word "position." The mathematics teacher, somewhat indignant but more determined than ever, used a lead pencil to mark a point P on a piece of paper (Figure 2–3). She considered this lead pencil mark to be a position on the piece of paper. She then folded another piece of paper and used it as a straight edge to draw a line through point P; that is, through the position indicated by P (Figure 2–4). This line contained infinitely many points. The astronomer remarked how difficult it was to locate the original point P of the line without another line through that point (Figure 2–5). The astronomer thought of a point on the paper as the intersection of two lines.

Under the mathematics teacher's interpretation of a point as a position on a sheet of paper, and indicated by the point of intersection of two lines, the group could describe the corners of the chessboard since they represented the intersections of lines along the side of the chessboard. Consider the dashed lines in Figure 2–6.

Everyone in the room seemed satisfied with the new concept of position except the piano player, who from the background of his own world of music decided to ask the professor another question.

"You seem to be way out in space most of the time. Can you tell me *where* is the edge of space?" Again all eyes were fixed on the professor eagerly awaiting his reply. He did not answer them, but instead, asked, "What do you mean by the edge of space?"

At lunch the mathematics teacher looked at the edge of the table shown in Figure

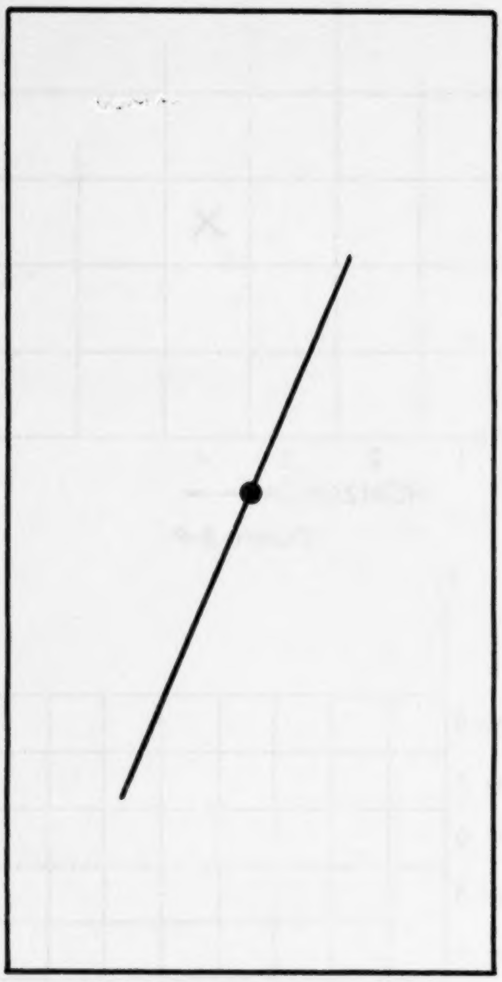


Figure 2–4

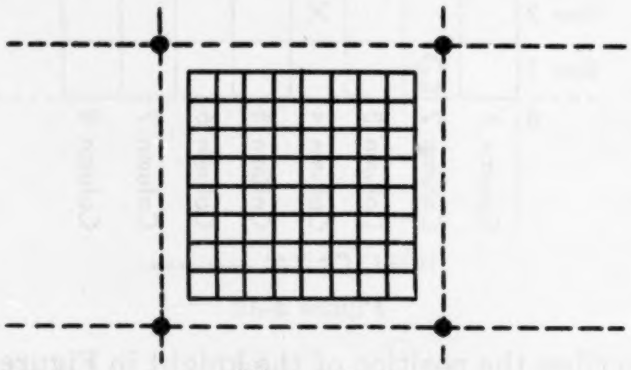


Figure 2–6

2–7 and thought, "The edge of the table is the intersection of two planes!" She was thinking of the set of points in the line formed by the fold in the tablecloth as it hung over the edge of the table. In her mind she realized why the professor asked *what* is meant by the edge of space. The edge of space might possibly be located using a system of coordinates similar to those on a chessboard. Perhaps this system can be used to answer the question *"Where do you live?"*

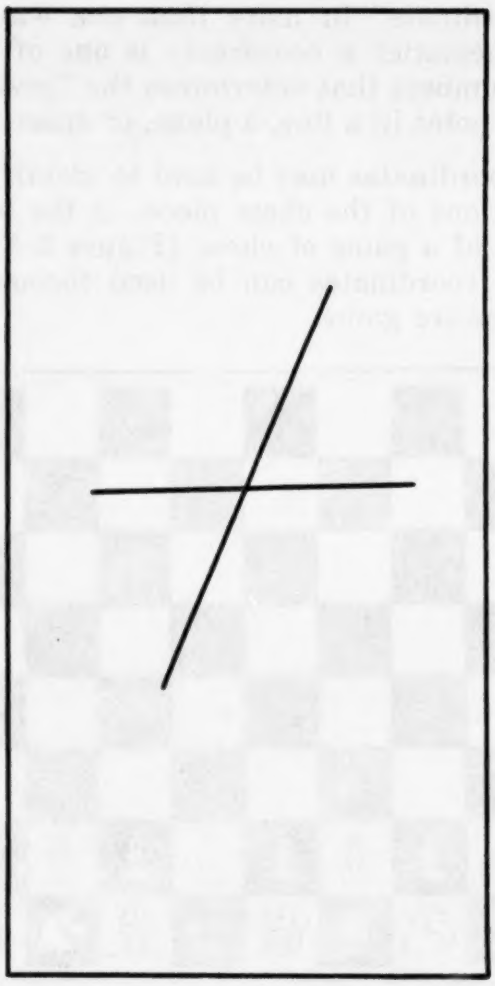


Figure 2–5

Coordinate (co-or′di-nate) systems have many applications here on the earth as well as in "outer space." Before applying coordinates to your own situation, you should first learn the meaning of a coordinate. Most dictionaries interpret the word

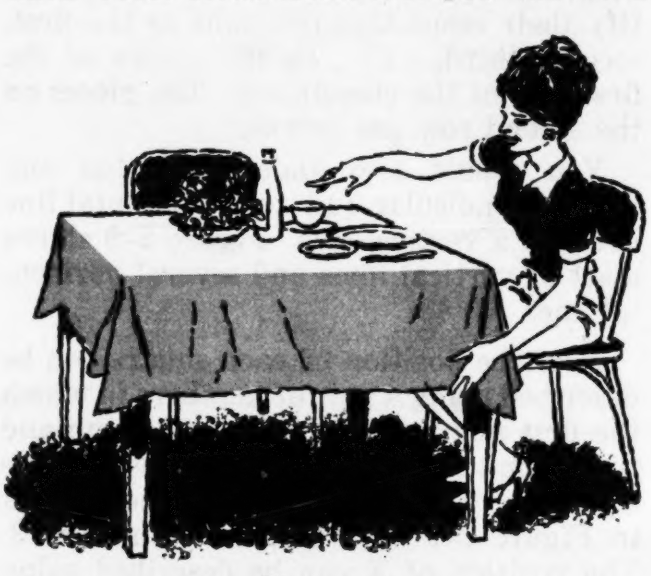


Figure 2–7

"coordinate" in more than one way. In mathematics a *coordinate* is one of a set of numbers that determines the "position" of a point in a line, a plane, or space.

Coordinates may be used to identify the positions of the chess pieces at the beginning of a game of chess (Figure 2–8). In fact, coordinates can be used throughout the entire game.

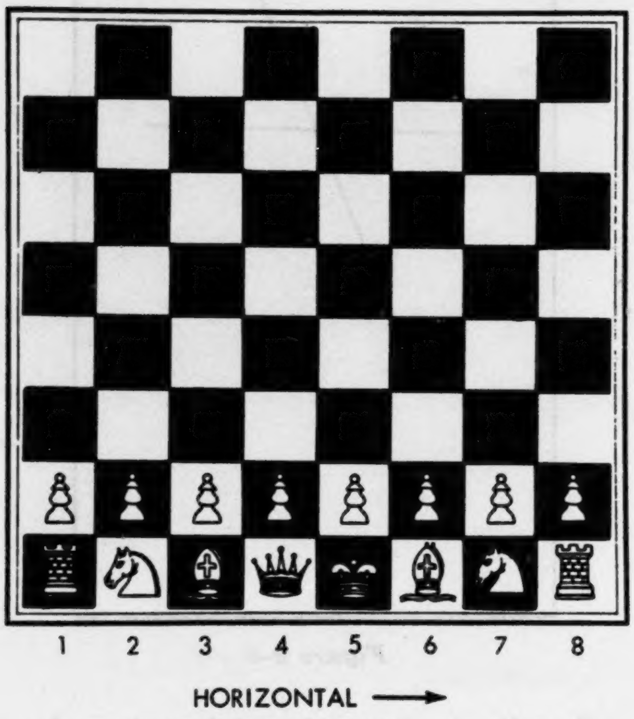


Figure 2–8

The chess pieces in the first row of Figure 2–8 may be identified from left to right as a castle, knight, bishop, queen, king, bishop, knight, castle. We can identify their respective positions as the first, second, third, . . . , eighth square of the first row of the chessboard. The pieces on the second row are pawns.

You should also understand that any line perpendicular (⊥) to a horizontal line is called a vertical line. Figure 2–9 shows a set of vertical lines and several horizontal lines.

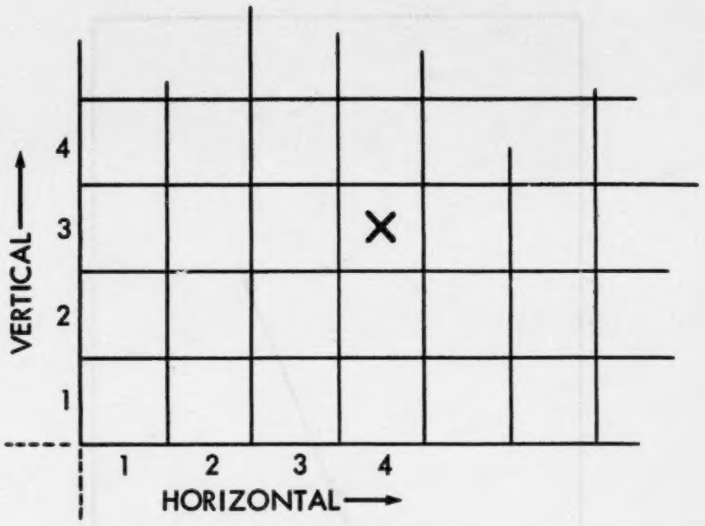


Figure 2–9

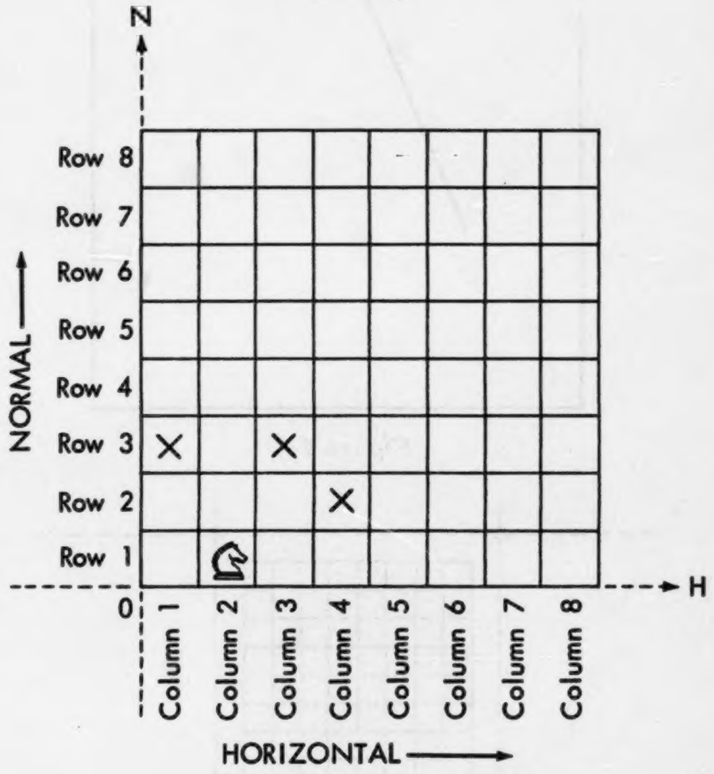


Figure 2–10

Now the position of each square can be described using a pair of numbers in which the first number identifies the column and the second number the row in which the square is located. For example, the X mark in Figure 2–9 is in column 4 and row 3. The position of X can be described using the ordered pair (4, 3). Then (2, 1) describes the position of the knight in Figure 2–10 since the knight is in the second column and first row. Each "X" marks a position to which the knight can be moved. Each chess piece can move in a manner specified by the rules of the game.

The location of points or positions frequently involves the use of coordinate systems similar to that used for the chessboard. Coordinate systems may be used in the description of geometric figures by identifying the positions of points such as A, B, and C, which determine the right triangle in Figure 2–11. Can you describe

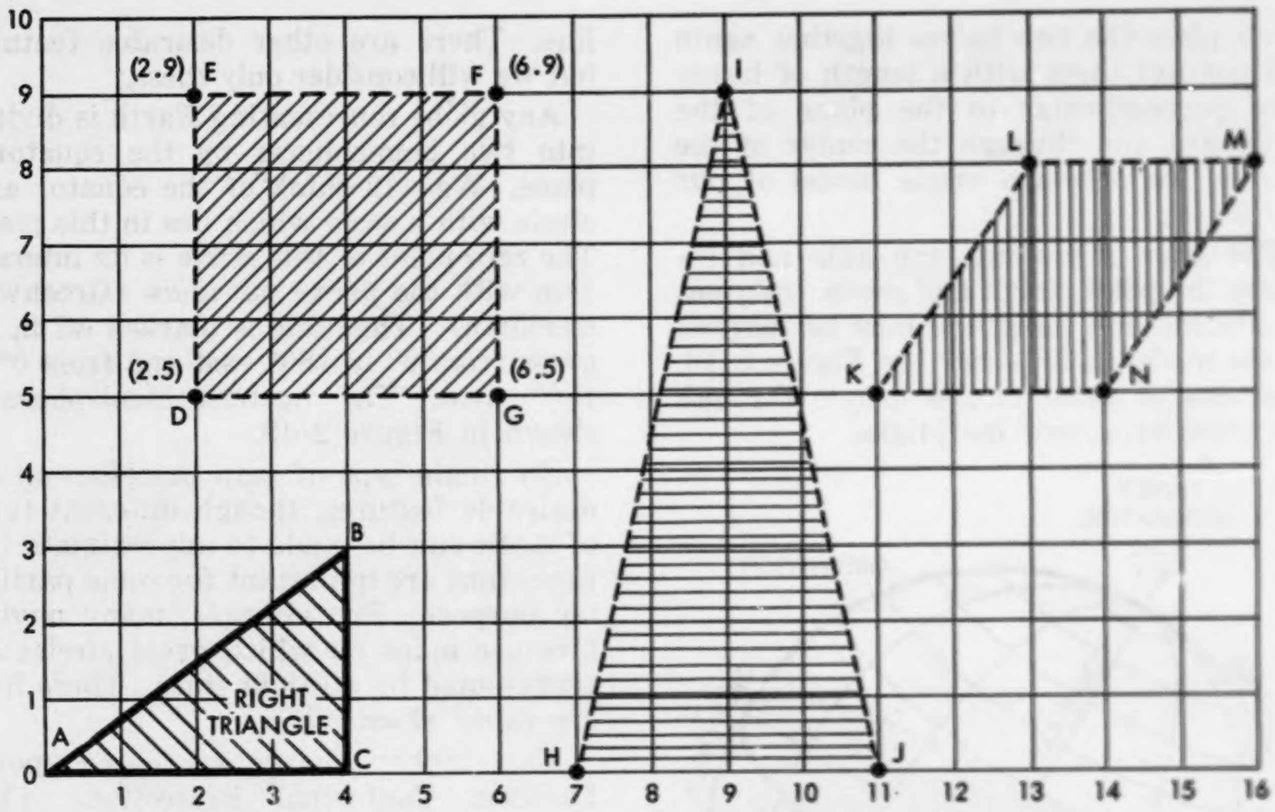


Figure 2-11

the other geometric figures by listing the coordinates of the points given on the chart?

Latitude and longitude for locating positions on Earth provide another example of a coordinate system. Other coordinate systems are used for locating points in space.

2-1 Exercises—The Universe We Live In

1. Locate and identify 10 lines in your room as intersections of planes.
2. Are there any points in your room equidistant from two fixed points located in your room? Can you describe their locations?
3. How could a blind man in Canada play chess with someone in the United States? (Perhaps you have a pen pal who enjoys chess?).

2–2. Relative Positions on Earth

Many people today are aware of the description of Earth as "pear-shaped." For our purposes Earth may be considered to be spherical. The geographer often discusses positions on Earth in terms of longitude and latitude (Section 1–4). This sometimes causes confusion since there are only two coordinates and two coordinates suggest a plane surface to many people instead of the surface of a solid.

If an orange is sliced in half and a cardboard is placed between the two halves, the cardboard may then be compared to the equatorial plane of Earth (Figure 2–12).

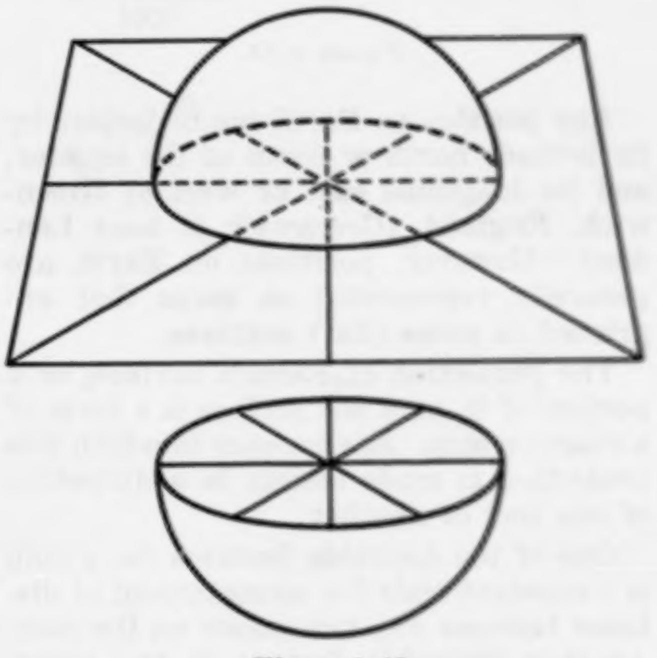


Figure 2-12

If we place the two halves together again and connect them with a length of heavy wire perpendicular to the plane of the cardboard and through the center of the orange, we obtain a crude model of our Earth.

The wire represents the axis and includes the poles, north and south. Degrees of latitude and longitude may be marked on the model as illustrated in Figure 2–13. The arcs of great circles drawn through the poles represent meridians.

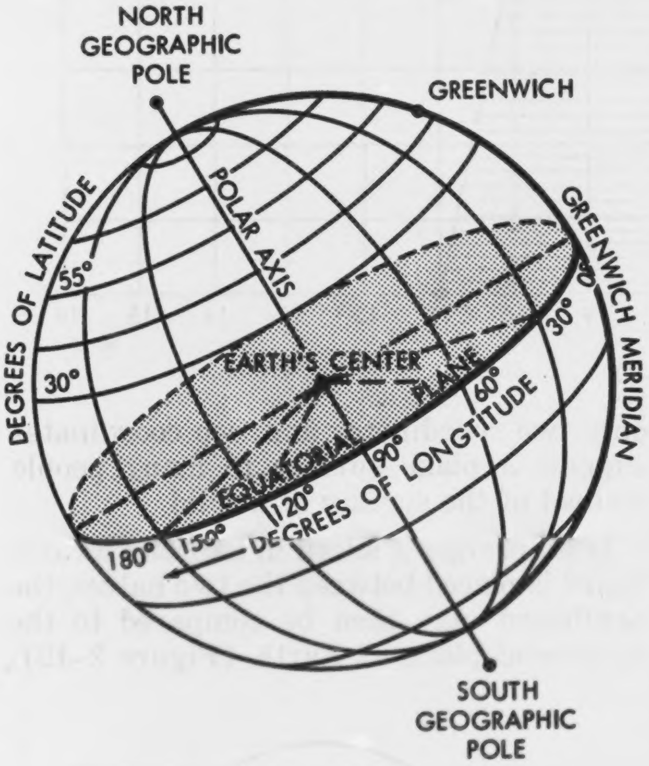


Figure 2–13

Any position on Earth can be located by its latitude north or south of the equator, and its longitude east or west of Greenwich, England (Greenwich is near London). However, positions on Earth are generally represented on maps that are printed on plane (flat) surfaces.

The projection of Earth's surface, or a portion of it, on a flat surface is a form of a chart or map. Any manner in which this projection is made results in a distortion of one sort or another.

One of the desirable features for a map is a constant scale for measurement of distance between any two points on the map. Another desirable feature is the representation of a great circle as a straight line. There are other desirable features but we will consider only these.

Any globe representing Earth is divided into two hemispheres by the equatorial plane. We will think of the equator as a circle with a scale which lies in this plane. The zero point of this circle is its intersection with the *prime meridian* (Greenwich meridian). The scale is marked off in degrees from 0° to 180° east and from 0° to 180° west. The western hemisphere is shown in Figure 2–13.

No single type of map possesses all the desirable features, though different types of maps can be made to approximate features that are important for some particular purpose. For example, many navigators use maps on which great circles are represented by straight lines. These lines are called *rhumb lines.*

Most highway maps are based upon a Lambert Conformal Projection. (You should obtain an ordinary highway map of the United States in order to best follow the discussion and exercises of the next few paragraphs.)

TABLE 2–1 Cities of the U.S.

Albuquerque, New Mexico
Oklahoma City, Oklahoma
Santa Ana, California
Omaha, Nebraska
Tulsa, Oklahoma
Madison, Wisconsin
Springfield, Illinois
Detroit, Michigan
New York, New York
Tampa, Florida
New Orleans, Louisiana
Jackson, Mississippi
Memphis, Tennessee
Atlanta, Georgia
Yorktown, Virginia
Chicago, Illinois
Myrtle Beach, South Carolina
Burlington, Vermont

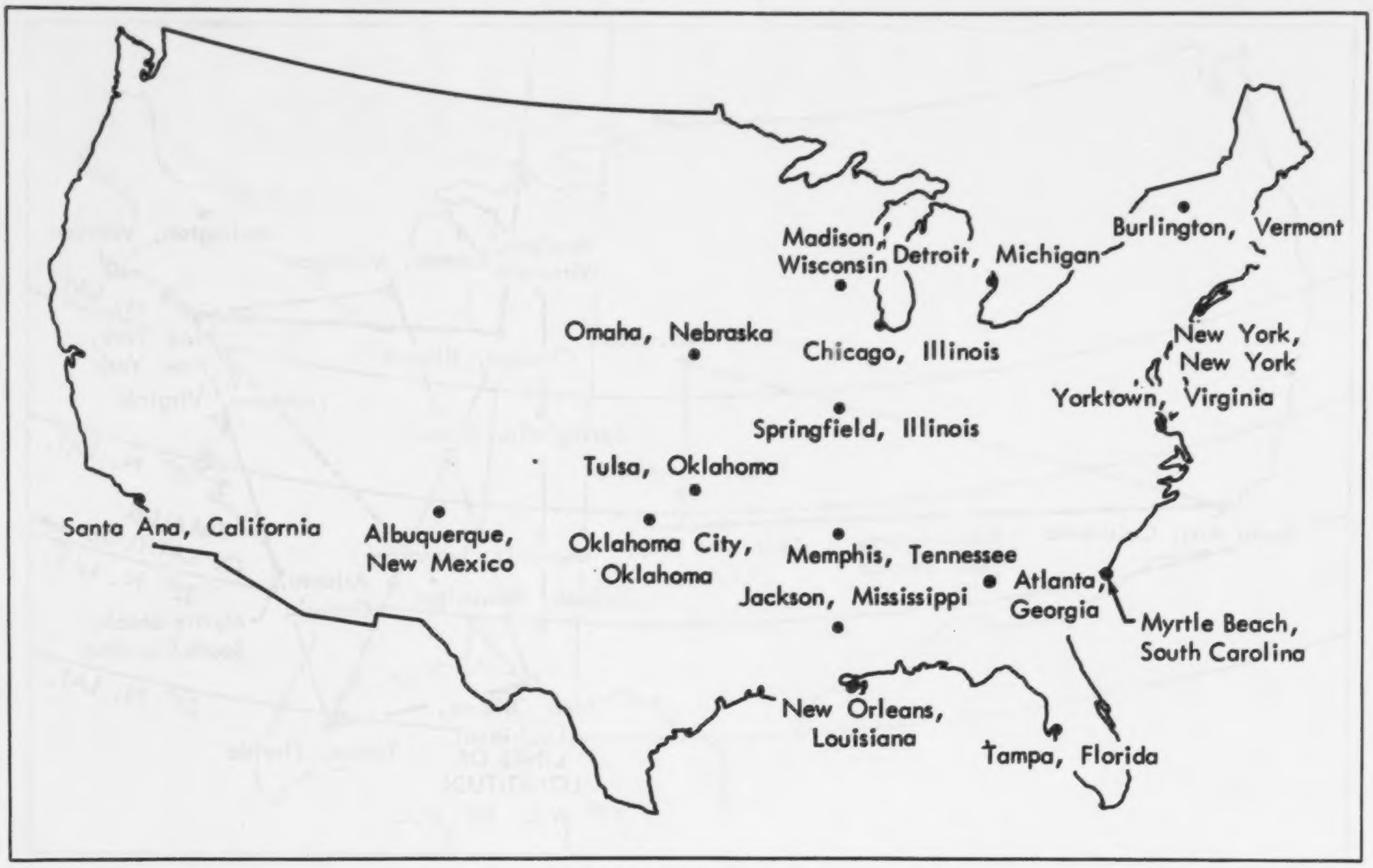

Madison, Wisconsin
Detroit, Michigan
Burlington, Vermont
Omaha, Nebraska
Chicago, Illinois
New York, New York
Yorktown, Virginia
Springfield, Illinois
Tulsa, Oklahoma
Santa Ana, California
Albuquerque, New Mexico
Oklahoma City, Oklahoma
Memphis, Tennessee
Atlanta, Georgia
Jackson, Mississippi
Myrtle Beach, South Carolina
New Orleans, Louisiana
Tampa, Florida

Figure 2–14

Use a marking pen to circle on the map the position of each of the cities listed in Table 2–1 as shown in Figure 2–14.

Now use a straight edge (such as a yard stick) to draw the following lines. (Every effort should be made to choose the straight line which best fits the points representing cities that have been circled.)

1. A line through Omaha and Tulsa.
2. A line through Madison and Jackson.
3. A line through Santa Ana and Oklahoma City.
4. A dotted line through Omaha and Santa Ana.
5. A line through Chicago and Tampa.
6. A line through New York and Tampa.
7. A line through Chicago and New York.
8. A dotted line through Tampa and New Orleans.
9. A dotted line through New Orleans and New York.
10. A dotted line through Detroit and New York.

Your map should now resemble Figure 2–15.

The straight lines resemble air line routes from city to city. There appears to be a straight line from Madison, Wisconsin, through Springfield, Illinois, through Memphis, Tennessee, to Jackson, Mississippi. The line that passes near Santa Ana, California, Albuquerque, New Mexico, and Oklahoma City, Oklahoma, appears to be perpendicular to the line through Omaha, Nebraska, and Tulsa, Oklahoma. There appears to be an isosceles triangle with vertices at Chicago, Illinois; New York, New York; and Tampa, Florida. Use a protractor and straight edge to verify these conjectures. One may also conjecture a parallelogram with vertices at the cities of New Orleans, Tampa, New York, and Detroit. What do you know that can help you to determine whether this conjecture is true or false?

Compare the data in Table 2–2 to your own observations of longitude and latitude using an ordinary highway map.

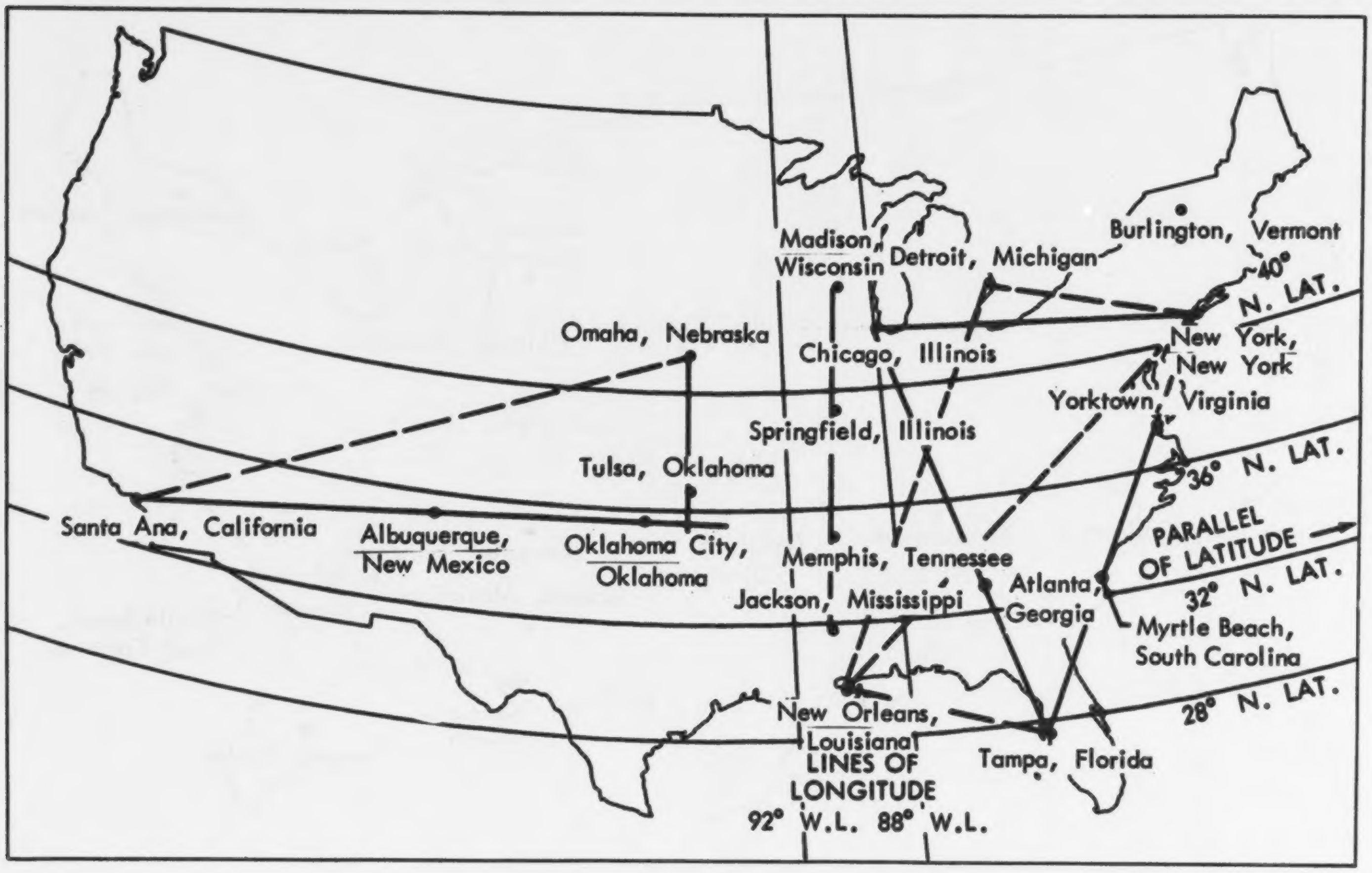

Burlington, Vermont
Madison, Wisconsin
Detroit, Michigan
40° N. LAT.
New York, New York
Omaha, Nebraska
Chicago, Illinois
Yorktown, Virginia
Springfield, Illinois
Tulsa, Oklahoma
36° N. LAT.
Santa Ana, California
Albuquerque, New Mexico
Oklahoma City, Oklahoma
Memphis, Tennessee
PARALLEL OF LATITUDE
Atlanta, Georgia
32° N. LAT.
Jackson, Mississippi
Myrtle Beach, South Carolina
New Orleans, Louisiana
28° N. LAT.
LINES OF LONGITUDE
Tampa, Florida
92° W.L.
88° W.L.

Figure 2-15

TABLE 2-2

Approximate Longitude and Latitude
Selected Cities of the U.S.

CITY	LONGITUDE	LATITUDE
Madison, Wisconsin	89° 23′ W	43° 7′ N
Springfield, Illinois	89° 44′ W	39° 51′ N
Detroit, Michigan	83° 0′ W	42° 18′ N
New York, New York	71° 50′ W	40° 45′ N
Tampa, Florida	82° 30′ W	28° 0′ N
New Orleans, Louisiana	90° 5′ W	29° 57′ N
Jackson, Mississippi	90° 12′ W	32° 19′ N
Memphis, Tennessee	90° 0′ W	35° 9′ N
Atlanta, Georgia	84° 23′ W	33° 45′ N
Yorktown, Virginia	76° 30′ W	37° 14′ N
Chicago, Illinois	87° 38′ W	42° 12′ N
Myrtle Beach, South Carolina	79° 50′ W	33° 45′ N
Burlington, Vermont	73° 15′ W	44° 30′ N

You have probably observed that you cannot rely on a highway map as a true representation of the positions of points on the surface of the Earth. There are frequently noticeable errors to be found. Measure the distance from Chicago, Illinois to New York, New York, using the scale on a map of the United States. Compare this scale measurement to the actual distance as listed in a table of distances usually provided on one corner of the map.

2–2 Exercises Relative Position on Earth

1. On a highway map, what city best represents the intersection of the line through Omaha, Nebraska and Tulsa, Oklahoma with the line through Santa Ana, California and Oklahoma City, Oklahoma?
2. What is the approximate longitude and latitude of the city at the intersection mentioned in Exercise 1?
3. Use a highway map and approximate the number of miles which may be saved by flying directly to Omaha, Nebraska from Santa Ana, California instead of going east to Henryetta, Oklahoma and then north to Omaha, Nebraska.
4. Is the figure formed on a highway map by connecting the vertices at the cities of New Orleans, Tampa, New York, and Detroit a parallelogram?

2–3 Fallacies of maps

The points on a map of a part of Earth represent positions on the surface. How are maps of the surface of a solid obtained on a plane surface? You can transfer a design from a cylindrical roller onto a flat surface as is done by many painters and some printing presses. But how can you map the surface of a sphere onto a plane surface? We have already observed that there will be some distortions. We now consider the problem further in terms of locating your *zenith*.

First you need to find a point directly above you. How would you do this? An empirical method of determining a zenith point is in the form of a game in which any number of players may participate. The object of the game is to point to the point on the ceiling that represents the zenith of a point on the floor.

The materials needed for the game are:

1 roll of tape
1 straight pin
10 feet of string
1 sharpened pencil
1 piece wrapping paper or cardboard (about 4 feet square)
1 straight edge (meter stick)
1 pointer (such as a straightened coat hanger)
1 stepladder
1 weight (such as a lead fishing sinker)
1 protractor

The procedure is to draw a circle on the cardboard or wrapping paper. The circle should have a diameter of about 60 centimeters. Draw and label a diameter IG with a midpoint X, the center of the circle. A string with a pin at one end and a pencil at the other may be used to draw the circle.

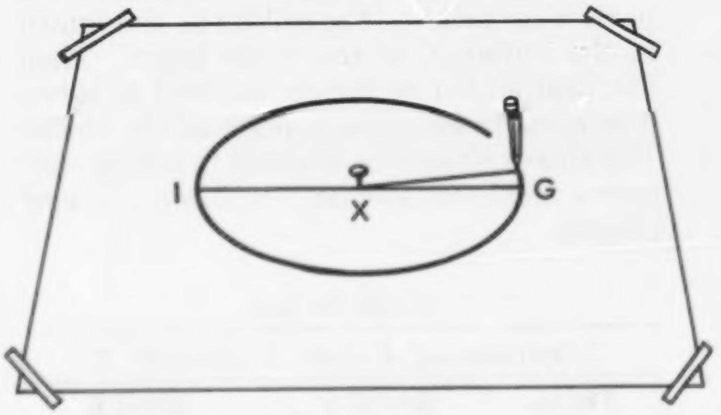


Figure 2–16

Tape one end of a piece of string to one end of the pointer. Now tie the lead sinker to the other end of the string making the length such that the lead sinker is about ½" above the floor when the tip of the pointer touches the ceiling (Figure 2–17).

Tape the cardboard or wrapping paper onto the floor with the circle and its center X clearly visible to all participants. Another piece of wrapping paper may be taped to the ceiling over the circle on the floor.

The object of the game is to use the pointer to touch the point on the ceiling directly above the center of the circle on the floor. The lead sinker is to be held off

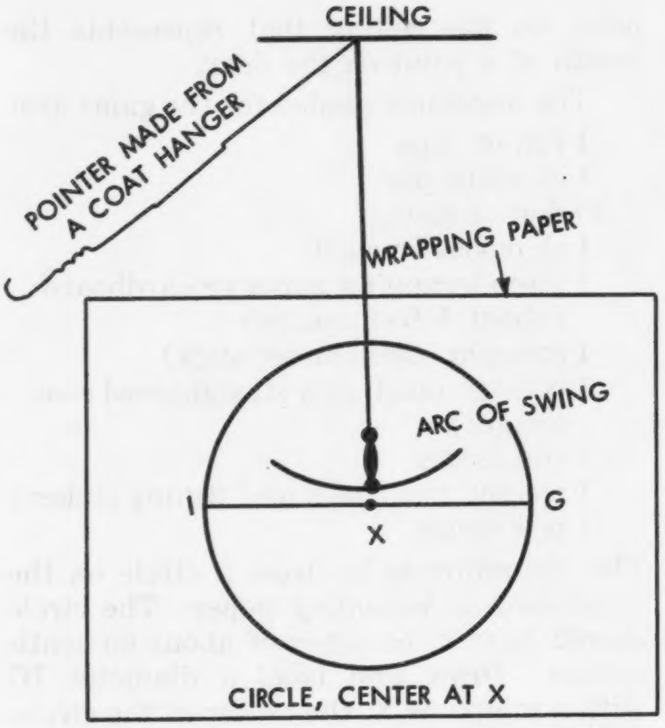


Figure 2-17

to one side by another person until the person doing the pointing has decided on a particular point of the ceiling as the zenith of the center X of the circle below. Then the lead sinker is slowly lowered to allow it to come to rest over a point of the circle. The sinker should be allowed to swing only over a small arc so that it will come to rest shortly.

TABLE 2-3

Locations of Points A through F

TRIAL	RADIUS	ANGLE
A	25 cm	+ 15°
B	24 cm	+ 30°
C	25 cm	+ 75°
D	12 cm	+168°
E	20 cm	− 90°
F	46 cm	−135°

Table 2-3 shows the listing of the results obtained in six trials. Notice that for each trial the position is recorded for the point directly under the one pointed to on the ceiling. These records are in terms of the distance from the center X and a direction. To provide a basis for this system of coordinates, the circle may be marked off in 15° intervals, then the line segments joining these points to the center should be marked off in centimeters as in Figure 2-18 where one unit represents five centimeters.

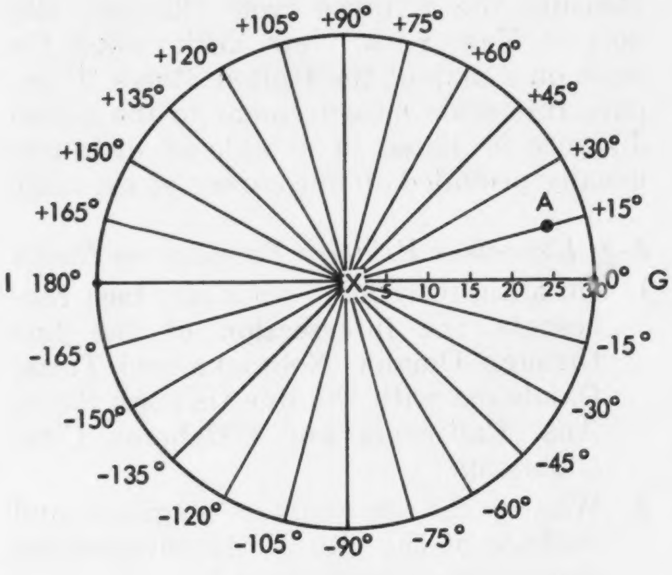


Figure 2-18

Point A represents the result of the first trial. The lead sinker came to rest over the point marked A which is 25 centimeters from X and +15° from the reference ray XG.

The game can be played using only the distance from the center as the criteria for judging accuracy. A more interesting approach is to use both the distance from the center and also the measure of the angle in degrees away from XG.

The remaining trials in Table 2-3 should be completed giving values of radii to the nearest tenth of a centimeter and the angles to the nearest degree.

The results of one game are listed in Table 2-4 as obtained by a family of four. The column containing trials identify each member's attempt.

F-1 represents a first attempt by the father.

M-2 represents the second attempt by the mother.

K-3 represents the third attempt by a child named Kim.

C-4 represents the fourth attempt by another child named Chris.

By examining the data, can you determine who most nearly succeeded in pointing to the zenith of a point on the floor?

TABLE 2–4

Sample of Trials by a Family

TRIAL	RADIUS	ANGLE
F–1	6.8 cm.	109°
M–2	13.0 cm.	0°
K–3	28.0 cm.	−178°
C–4	7.0 cm.	109°
K–5	28.5 cm.	87°
C–6	3.5 cm.	0°
M–7	17.9 cm.	91°
K–8	22.2 cm.	− 99°
C–9	6.4 cm.	108°
M–10	2.2 cm.	123°
K–11	18.7 cm.	94°
C–12	4.9 cm.	149°
M–13	3.8 cm.	111°
K–14	11.2 cm.	− 84°
C–15	7.4 cm.	108°
M–16	4.7 cm.	− 74°

Which member of the family picked a point directly over the ray XG?

Who was furthest from the center in centimeters? Who was furthest from the radius XG in degrees?

Figures 2–18 and 2–19 may now be compared to the screen of a radar scope used in tracking objects in space. The points B, C, D, etc., could resemble the positions of ships at sea or even points in space. The distances could represent the distance of the points from a fixed point on Earth and the angles represent the direction.

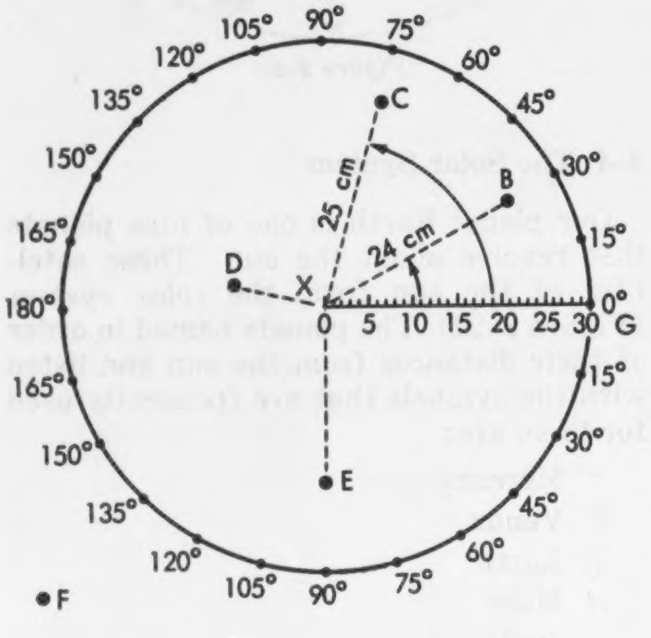


Figure 2–19

A globe is the most accurate scale model of Earth. It is the one place where you can find global relationships shown almost as they actually exist.

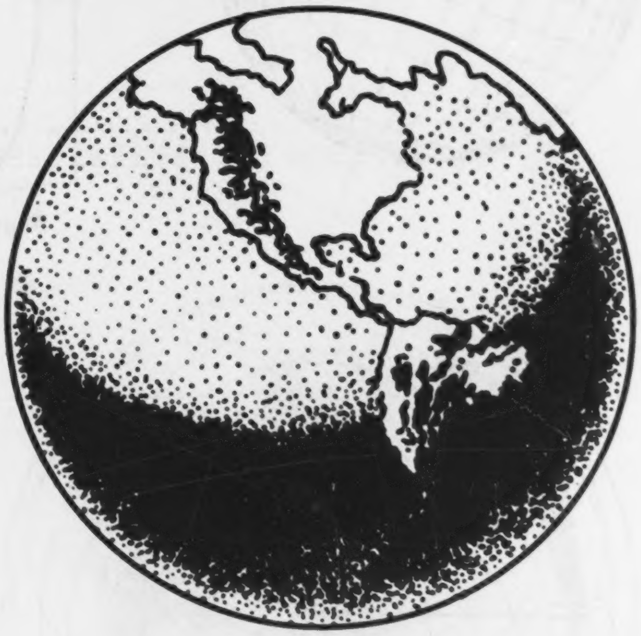


Figure 2–20

The globe has disadvantages in that it is difficult to construct, bulky, unwieldy, and one can see at most half of the surface at one time (Figure 2–20). These disadvantages have caused man to devise schemes to project (map) points from the globe to points on a plane. Some of the common forms of projection are the

mercator projection
orthographic projection (Figure 2–21)
azimuthal equidistant projection
(Figure 2–22)
gnomonic projection
cylindrical equal area projection
conic projection

There are also many other forms which may be investigated (Figure 2–23). Each of the different types of projections has advantages and disadvantages. Distortions arise in each projection. The problem is to choose the best projection for a given situation. We have now learned that we

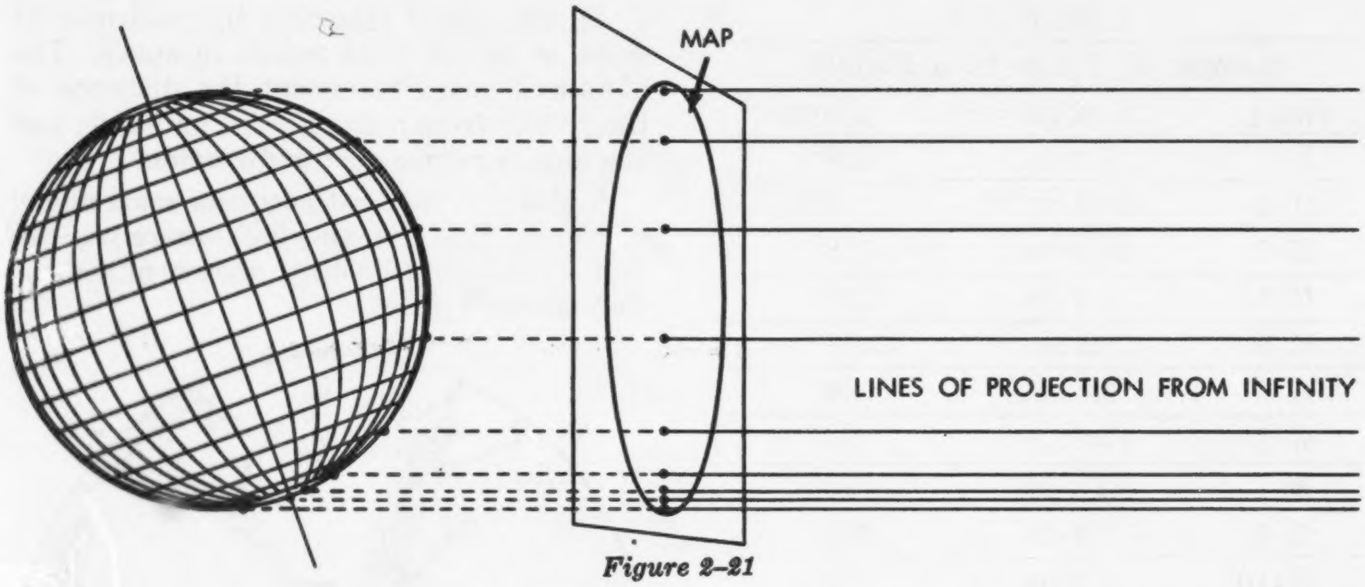

MAP
LINES OF PROJECTION FROM INFINITY

Figure 2–21

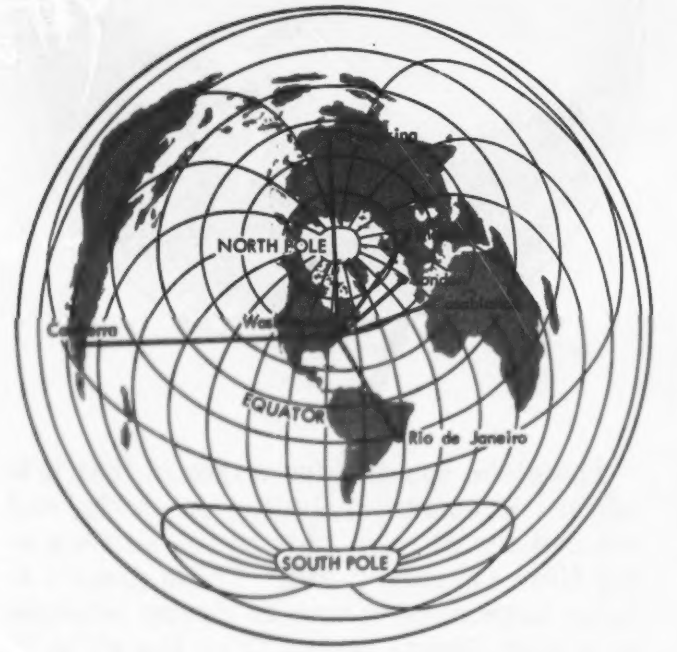

NORTH POLE
EQUATOR
Rio de Janeiro
SOUTH POLE

Figure 2–22

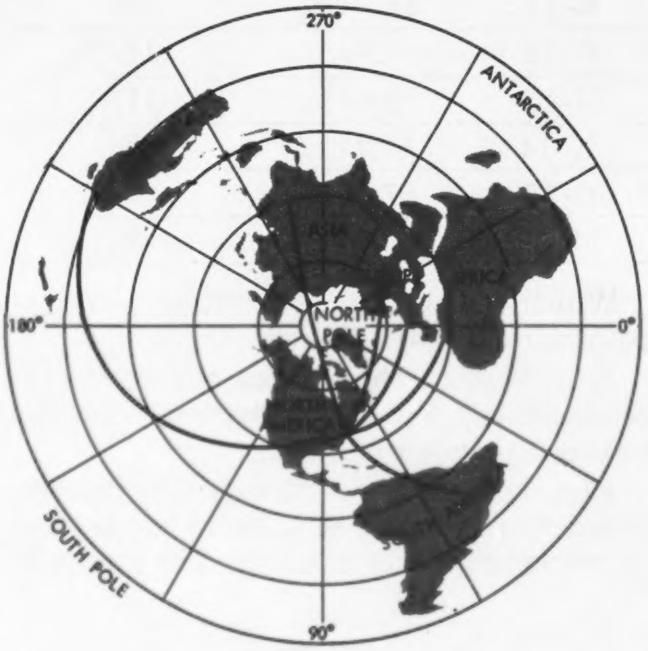

270°
ANTARCTICA
ASIA
180°
NORTH POLE
0°
SOUTH POLE
90°

Figure 2–23

must use different maps for different purposes due to the fallacies present in all flat maps of Earth.

2–3 Exercise—Fallacies of Maps

Copy Figure 2–19, extend XG to obtain a diameter of the circle and construct the diameter that is perpendicular to XG. Start at G and label the quadrants (quarters of the circular region) counter clockwise I, II, II, IV. Then identify the quadrant of the point for each trial in Table 2–4 except M–2 and C–6. (These two are on the common boundaries of the first and fourth quadrants.)

2–4 The Solar System

Our planet Earth is one of nine planets that revolve about the sun. These satellites of the sun form the *solar system* (Figure 2–24). The planets named in order of their distances from the sun and listed with the symbols that are frequently used for them are:

☿ Mercury
♀ Venus
⊕ Earth
♂ Mars
♃ Jupiter

♄ Saturn
♅ Uranus
♆ Neptune
♇ Pluto

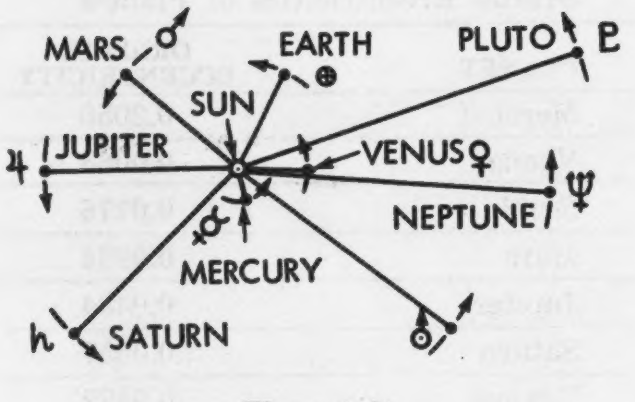

MARS ♂
EARTH ⊕
PLUTO ♇
SUN
♃ JUPITER
VENUS ♀
♆ NEPTUNE
♅
MERCURY ☿
♄ SATURN

Figure 2–24

The representation in Figure 2–24 is distorted since the orbits are not actually circles and also the orbits are not all in the same plane. We may however think of the planets as revolving counterclockwise (relative to the view shown in Figure 2–24) about the sun is elliptical orbits which are nearly in the same plane.

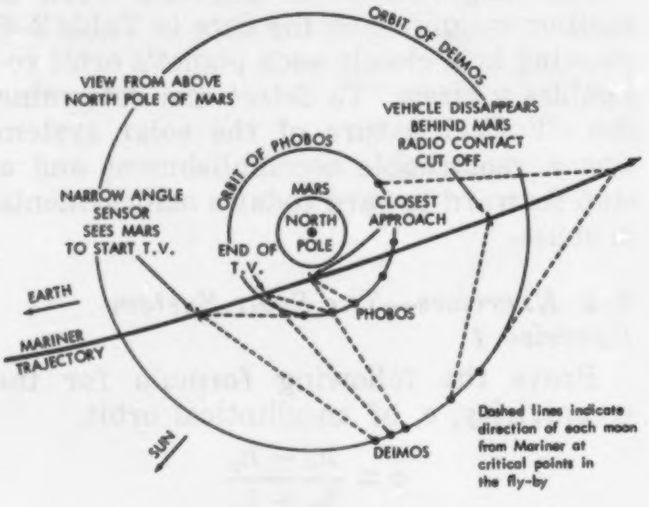

ORBIT OF DEIMOS
VIEW FROM ABOVE NORTH POLE OF MARS
VEHICLE DISSAPPEARS BEHIND MARS RADIO CONTACT CUT OFF
ORBIT OF PHOBOS
NARROW ANGLE SENSOR SEES MARS TO START T.V.
MARS
NORTH POLE
CLOSEST APPROACH
END OF T.V.
EARTH
PHOBOS
MARINER TRAJECTORY
Dashed lines indicate direction of each moon from Mariner at critical points in the fly-by
DEIMOS
SUN

Figure 2–25

The illusion of objects moving in the same plane is very common and frequently depends upon the position of the observer. For example, in Figure 2–25 Mariner 4 appears to pass close to Mars in the plane of the orbits of Mars' satellites (moons) Phobos and Deimos; in Figure 2–26 we see that Mariner 4 actually dipped under the plane of the orbits of these satellites.

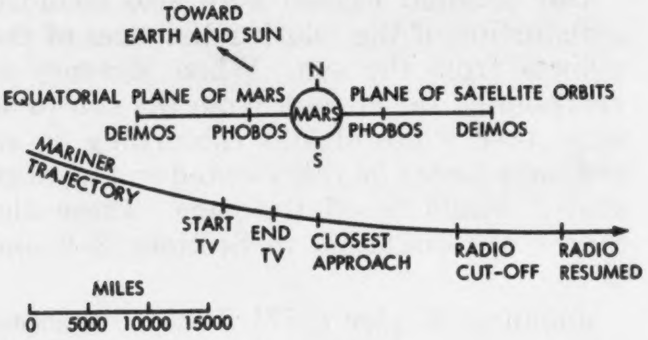

TOWARD EARTH AND SUN
N
EQUATORIAL PLANE OF MARS
PLANE OF SATELLITE ORBITS
MARS
DEIMOS
PHOBOS
PHOBOS
DEIMOS
S
MARINER TRAJECTORY
START TV
END TV
CLOSEST APPROACH
RADIO CUT-OFF
RADIO RESUMED
MILES
0
5000
10000
15000

Figure 2–26

If we take a different view of the solar system (Figure 2–27) we can see that the orbital planes of the planets are distinct. Think of the sun at the top of a pole with several spheres attached by strings and revolving about the top of the pole.

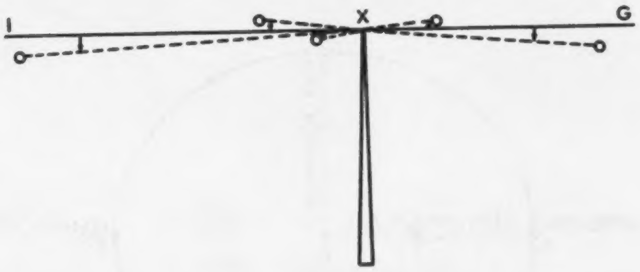


Figure 2–27

In Figure 2–27 the plane of Earth's orbit (the plane of *ecliptic*) is designated IG and is used as a reference plane. The plane of the orbit of each planet forms an angle with the plane of ecliptic. These angles are the *inclinations* of the orbits of the planets. (Table 2–5)

TABLE 2–5

PLANET	INCLINATION
Mercury	7° 0′
Venus	3° 24′
Earth	0° 0′
Mars	1° 51′
Jupiter	1° 18′
Saturn	2° 29′
Uranus	0° 46′
Neptune	1° 46′
Pluto	17° 09′

Our original Figure 2–24 also included a distortion of the relative distances of the planets from the sun. When Mercury is represented far enough from the sun to be seen, then Pluto should (according to an ordinary scale) be represented so far away that it would be off the page. These distances are discussed in Sections 2–9 and 2–10.

Johannes Kepler (1571–1630) developed a theory in which the orbits of the planets were elliptical (or as he called them, "eccentric circular orbits"). We shall use Earth's orbit to illustrate what is meant by an ellipse. This orbit is grossly exaggerated in Figure 2–28 where the orbit has center C, foci F and S, major axis AP, minor axis MN, focal distance FC, *aphelion* A, and *perihelion* B.

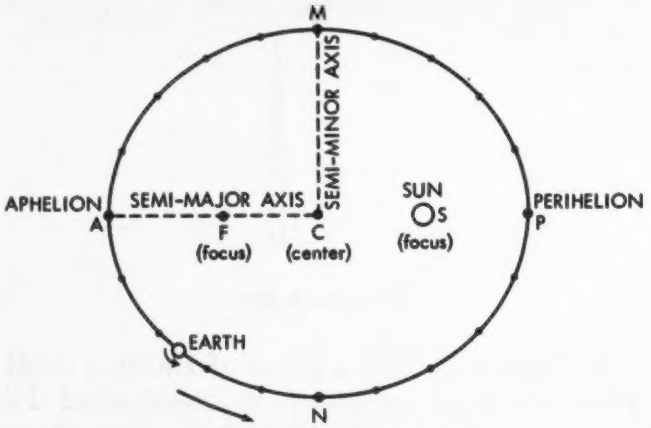


Figure 2–28

An *ellipse* is a simple closed plane curve such that the sum of the distances of each point from two given points *(focii)* is a constant (the length of the major axis). Notice that if the foci F and S coincided with the center C, the ellipse would be a circle of radius $\overline{AC}$. Thus the lengths of $\overline{FC}$ and $\overline{AC}$ may be used to indicate the extent to which the ellipse differs from a circle. The ratio $\frac{\overline{FC}}{\overline{AC}}$ is the *eccentricity* of the ellipse. The eccentricity is always less than 1. When the eccentricity is zero, we have a circle; when the eccentricity is approximately zero, we have approximately a circle.

The eccentricity of Earth's orbit is so small that, unless one is doing astronomical research, Earth's orbit is considered to be circular for all practical purposes. The eccentricities of the planets in the solar systems are listed in Table 2–6.

TABLE 2–6

Orbital Eccentricities of Planets

PLANET	ORBITAL ECCENTRICITY
Mercury	0.2056
Venus	0.0068
Earth	0.0176
Mars	0.0934
Jupiter	0.0484
Saturn	0.0557
Uranus	0.0472
Neptune	0.0086
Pluto	0.2502

Kepler solved an almost impossible problem using empirical data (that is, data obtained by experimentation and observation). The instruments used to obtain the data would be considered extremely crude and obsolete today.

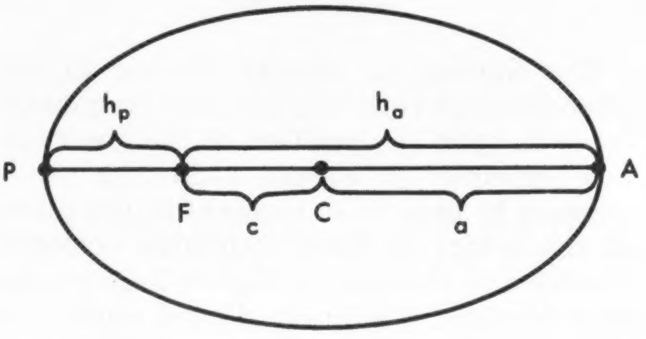
The magnificence of Kepler's work is further magnified by the data in Table 2–6 showing how closely each planet's orbit resembles a circle. To detect and determine the elliptical nature of the solar system was a remarkable accomplishment and a step forward toward today's achievements in space.

2–4 Exercises—The Solar System

Exercise 1

Prove the following formula for the eccentricity, e, of an elliptical orbit.

$$e = \frac{h_a - h_p}{h_a + h_p}$$



Given:

c = distance from center C to focus F.
a = length of semi-major axis $\overline{CA}$
e = eccentricity of the ellipse
h_a = height at apogee or aphelion
h_p = height at perigee or perihelion

Prove: $e = \frac{h_a - h_p}{h_a + h_p}$.

1. $\overline{PC} = \overline{AC} = a$
   C is the midpoint of $\overline{PA}$
2. $\overline{PC} = \overline{PF} + \overline{FC}$
   The whole is equal to the sum of its parts
3. $h_p = \overline{PF}$ and $c = \overline{FC}$
   Given
4. $h_p + c = a$
   Substitution Axiom
5. $h_p = a - c$
   Subtraction Axiom
6. $h_a = a + c$
   Given
7. $h_a - h_p = (a + c) - (a - c)$
   Subtraction Axiom
8. $h_a + h_p = (a + c) + (a - c)$
   Addition Axiom
9. $e = \frac{c}{a}$
   By definition in Section 2–4
10. $\frac{h_a - h_p}{h_a + h_p} = \frac{(a + c) - (a - c)}{(a + c) + (a - c)} = \frac{2c}{2a} = \frac{c}{a}$ Division Axiom
11. Since $e = \frac{c}{a}$ and $\frac{c}{a} = \frac{h_a - h_p}{h_a + h_p}$,
    then $e = \frac{h_a - h_p}{h_a + h_p}$ Transitive Axiom

Exercise 2

Compare the eccentricities of the following satellites.

Satellite	Apogee	Perigee	Eccentricity
Sputnik 1	588 miles	141 miles	?
Sputnik 2	1038 miles	140 miles	?
Explorer 1	1584 miles	224 miles	?
Vanguard 1	2462 miles	405 miles	?
Telstar 2	6712 miles	606 miles	?

Solution for Exercise 2

Sputnik 1

$$e = \frac{588 - 141}{588 + 141} = \frac{447}{729} = 0.613$$

Sputnik 2

$$e = \frac{1038 - 140}{1038 + 140} = \frac{898}{1178} = 0.762$$

Explorer 1

$$e = \frac{1584 - 224}{1584 + 224} = \frac{1360}{1808} = 0.752$$

Vanguard 1

$$e = \frac{2462 - 405}{2462 + 405} = \frac{2057}{2867} = 0.717$$

Telstar 2

$$e = \frac{6712 - 606}{6712 + 606} = \frac{6106}{7318} = 0.834$$

2–5 Earth—A Satellite With Satellites

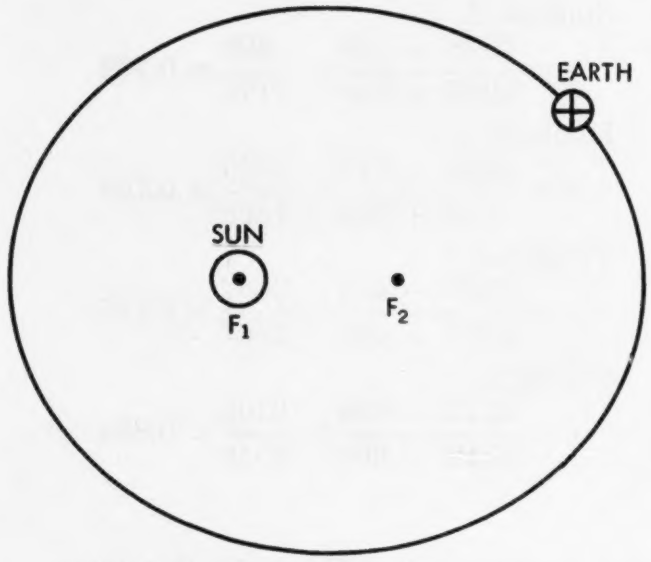
How can you tell that Earth is round and moves in an orbit about the sun? Children often accept these theories as truths told to them.

In Johannes Kepler's day, the leading teachers were professing statements contrary to today's modern theories concerning Earth and the solar system. In 1609, Johannes Kepler published "Commentaries on the Motion of Mars." In it he listed two unexplained facts which he deduced from the observations of Tycho Brahe. Nine years later in his book entitled, "The Harmony of the World," a third such fact was presented. All three facts have been accepted (postulated). These postulates became known as Kepler's Laws and mathematically described the orbits of the planets.

Kepler's Laws:

1. The orbit of each planet is an ellipse with the sun at one focus.
2. A line segment joining the sun and a planet, the radius vector, sweeps out equal areas in equal intervals of time.
3. The squares of the periods of revolutions of different planets around the sun are in the same proportion as the cubes of their mean distances from the sun.

Kepler's Laws are applicable not only to the orbits of planets but also to the orbit of the moon about Earth and to the orbits of manmade satellites.



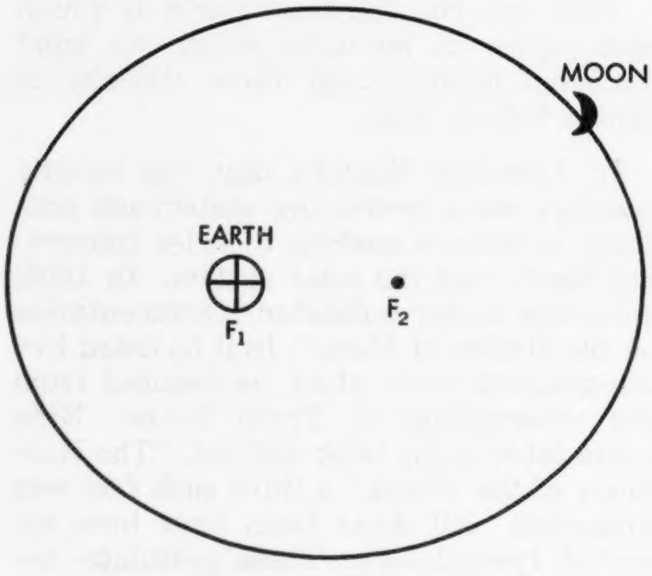


Figure 2–29

Both Earth's orbit about the sun and the moon's orbit about Earth are elliptical. Since Earth is at a focal point of the elliptical orbit of the moon, the distance of the moon from Earth varies. The point on the elliptical orbit of the moon that is nearest to Earth is called *perigee* and the point most distant from Earth is called *apogee* of the orbit.

The eccentricity of the moon's orbit about Earth is not constant. For this reason most textbooks do not list the eccentricity of the moon. Instead, an average value is sometimes listed and the explanation for the variations is discussed. The close association of the moon with Earth, combined with the tremendous gravitational influence of the sun, produces many changes *(perturbations)* in the orbit of the moon. These changes in turn are described by changes in the eccentricity of the orbit.

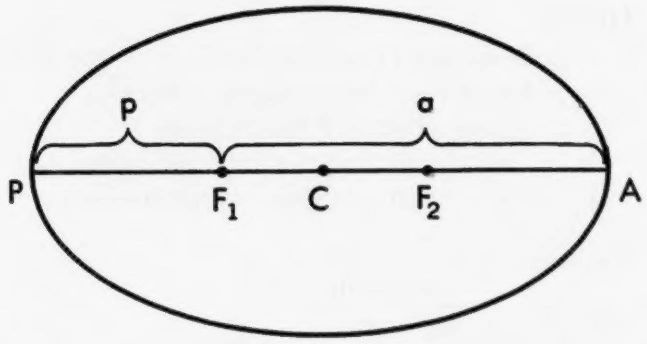


Figure 2–30

This same sort of perturbations (changes) exists in the orbits of manmade satellites of today. A further discussion of factors influencing the orbit of a satellite can be found in Section 5-8.

We shall limit our discussion of satellite orbits in this section to a theoretical model which undergoes no change in its circular orbit. Our theoretical model will be compared to Explorer XIV which had a period of 36.4 hours and came within 150 miles of the Earth. The orbit of Explorer XIV was inclined 33.1° to the equatorial plane of Earth.

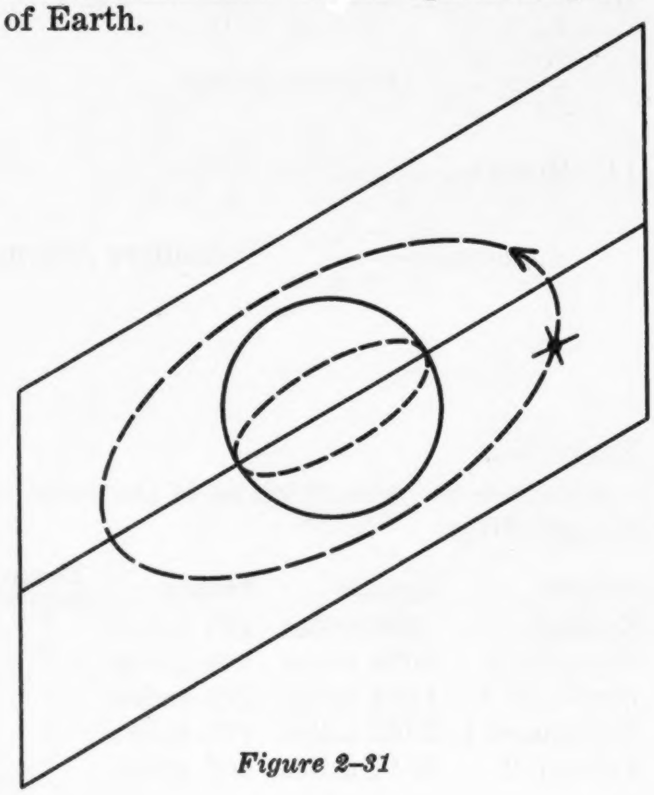


Figure 2–31

Consider a satellite having a period of 36 hours and a counterclockwise orbit in the equatorial plane of the Earth; that is, the inclination of the orbit is zero degrees (Figure 2–31). We shall consider ways of predicting the position of the satellite in its orbit. The prediction of the position of an Earth satellite is an important aspect of today's space age mathematics.

The following is an application of modular arithmetic and a student made "space-time" chart to predict the location of the satellite that we have described.

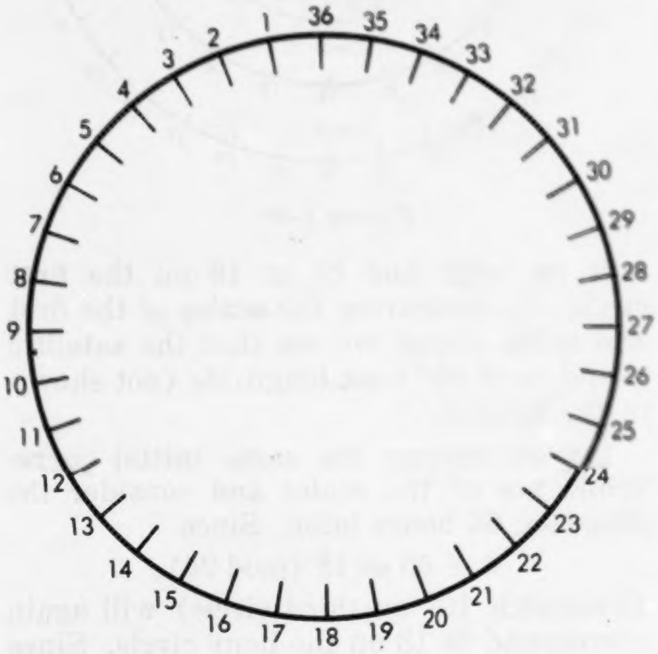


Figure 2–32

On a heavy sheet of paper or cardboard draw another circle of about 17 centimeters in diameter. Then as in Figure 2–32 mark off the circle in 10° intervals and label the marks from 1 to 36 counterclockwise. Call this circle and its scale the first circle. Since the satellite makes one revolution each 36 hours, we can identify its position on this scale (additional work will be needed to identify its position by the longitude of points on Earth). Suppose that the satellite starts at position 36 on our scale; then its position

10 hours later will be 10,
25 hours later will be 25,
36 hours later will be 36,
37 hours later will be 1,
45 hours later will be 9,
75 hours later will be 3,

and so forth. Notice that

$37 = 1 + 36,$
$45 = 9 + 36,$
$75 = 3 + 2\ (36);$

in other words,

37 and 1 differ by a multiple of 36
45 and 9 differ by a multiple of 36
75 and 3 differ by a multiple of 36

Whenever two numbers differ by an integral multiple of 36, the two numbers are *congruent modulo* 36. In the language of modular arithmetic we may write

$37 \equiv 1 \pmod{36}$
$45 \equiv 9 \pmod{36}$
$75 \equiv 3 \pmod{36}.$

Next draw concentric circles with diameters about 11 and 12 centimeters respectively; cut out the larger circle to obtain a 12 cm circular region; mark off the smaller circle in 15° intervals, and label from 1 to 24 counterclockwise for the hours of a day (Figure 2–33). Call this circle and its scale the second circle (or hour circle).

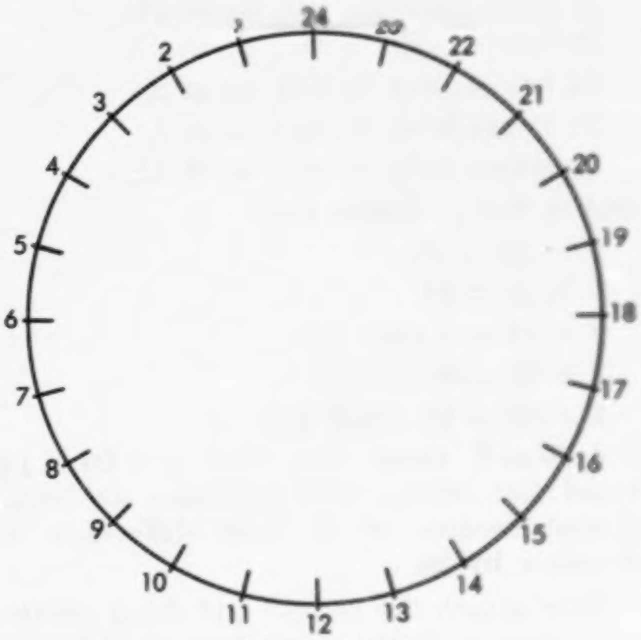


Figure 2–33

Finally, draw concentric circles with diameters about 6 and 7 centimeters; cut out the larger circle; mark off and label the smaller circle to represent the Greenwich meridian G, the international date line I, and degrees of east and west longitude (Figure 34). Call this circle and its scale the third circle.

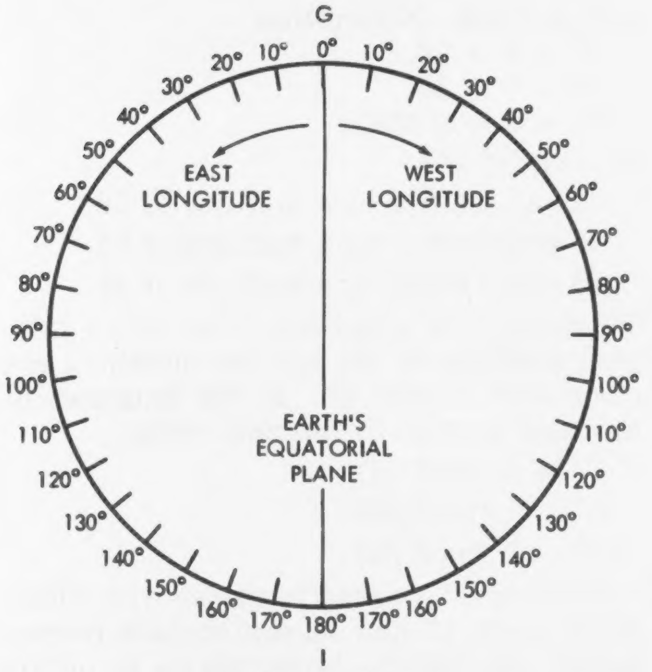


Figure 2–34

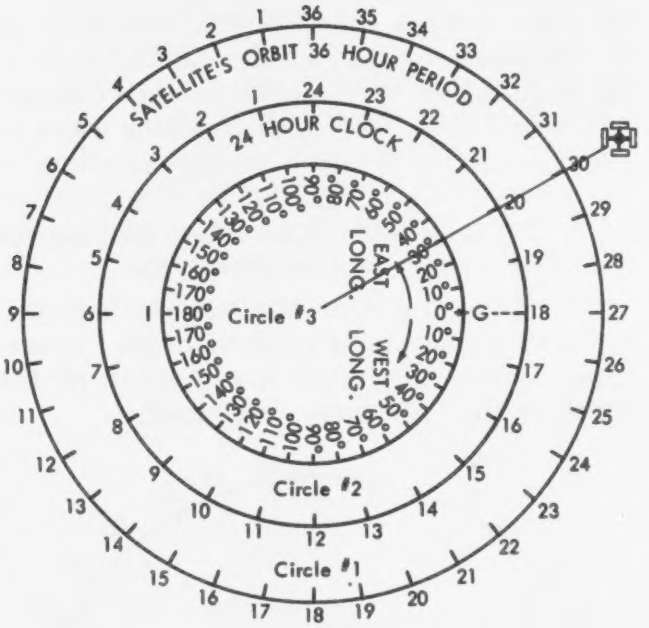


Figure 2–35

Use a straight pin to mount the third circle on and concentric with the second. Notice that if an observer at 0° longitude (Greenwich Meridian) starts at 2 on the second circle, then

15 hours later he will be at 17,
22 hours later he will be at 24,
24 hours later he will be at 2,
30 hours later he will be at 8,
60 hours later he will be at 14,

and so forth. Notice that

$2 + 15 = 17,$
$2 + 22 = 24,$
$2 + 24 \equiv 2 \pmod{24},$
$2 + 30 \equiv 8 \pmod{24},$
$2 + 60 \equiv 14 \pmod{24};$

$2 + 30 \equiv 8 \pmod{24}$, and $2 + 60 \equiv 14 \pmod{24}$ where two numbers are congruent modulo 24 if their difference is divisible by 24.

Now attach the second and third circles concentric with the first circle as in figure 2–35. Align the scales for an initial time with the satellite crossing the prime meridian; that is, align 36 on the first circle, 24 on the second circle, and 0° on the third circle (not shown in the figure). Eighteen hours later (keep the hour circle fixed and rotate the third circle counterclockwise) 0° corresponds to 18 on the hour circle (as shown in the figure). Also eighteen hours later the satellite would have made half an orbit and be at 18 on the first circle. By comparing the scales of the first and third circles we see that the satellite would be at 90° west longitude (not shown in the figure).

Let us assume the same initial correspondence of the scales and consider the situation 66 hours later. Since

$$0 + 66 \equiv 18 \pmod{24},$$

Greenwich (0° on third circle) will again correspond to 18 on the hour circle. Since

$$0 + 66 \equiv 30 \pmod{36},$$

the position of the satellite will correspond to 30 on the first circle. Then, as in Figure 2–35, the satellite will be at 30° east longitude.

We should also consider the problem of determining the portion of a satellite contained in a polar orbit (inclination of 90° to the equatorial plane) about Earth as in Figure 2–36. Notice that a polar orbit is in the plane of a meridian.

Our discussion will be limited to the model of a circular orbit and a perfect sphere to represent Earth. The procedure for determining the position (latitude) of a satellite in a polar orbit is very similar to that used to determine the position (longitude) for an equatorial orbit. We replace the third circle by a fourth circle marked to show degrees of latitude (Figure 2–37) and assemble the first, second, and fourth circles as in Figure 2–38. We

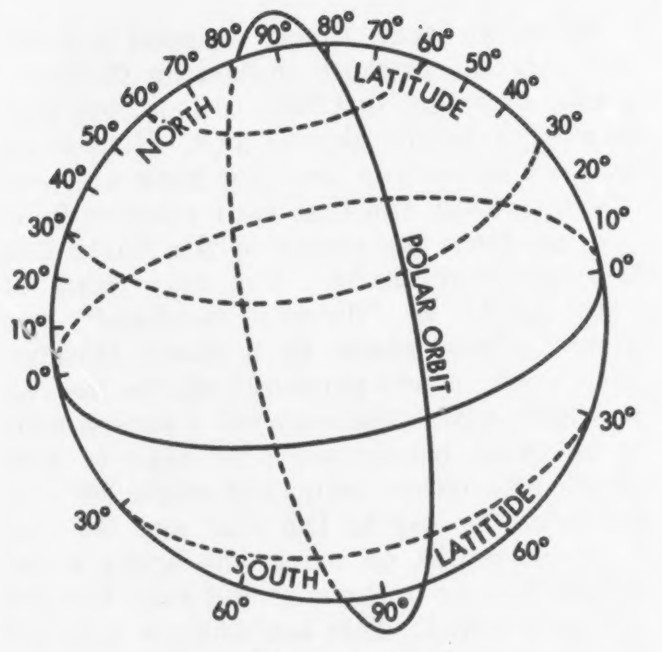

NORTH LATITUDE
POLAR ORBIT
SOUTH LATITUDE
90°
80°
70°
60°
50°
40°
30°
20°
10°
0°

Figure 2–36

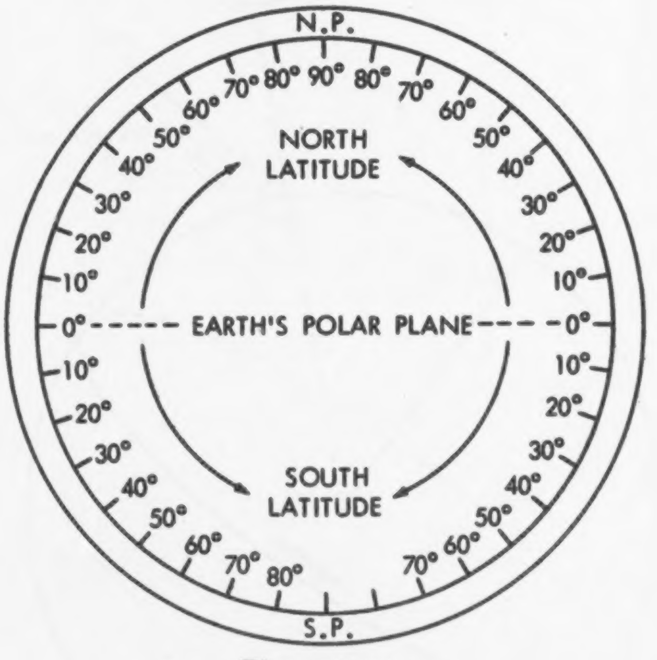

N.P.
NORTH LATITUDE
EARTH'S POLAR PLANE
SOUTH LATITUDE
S.P.
90°
80°
70°
60°
50°
40°
30°
20°
10°
0°

Figure 2–37

assume an initial alignment of the observer's latitude, 24 on the second circle, and position 36 for the satellite.

Suppose that an observer at 40° north latitude sees the satellite pass overhead, where will the satellite be 66 hours later? Since

$$0 + 66 \equiv 18 \pmod{24},$$

the observer's position (40° north latitude) on the fourth circle should be aligned with 18 on the hour circle. Since

$$0 + 66 \equiv 30 \pmod{36},$$

the position of the satellite corresponds to

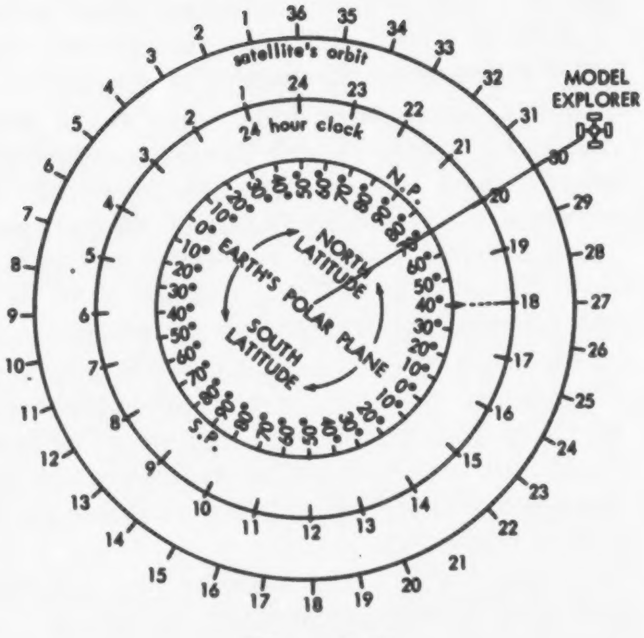

satellite's orbit
24 hour clock
MODEL EXPLORER
N.P.
NORTH LATITUDE
EARTH'S POLAR PLANE
SOUTH LATITUDE
S.P.

Figure 2–38

30 on the first circle. Then, as in Figure 2–38, the satellite is at 70° north latitude.

The procedures that we have considered may be used for circular, polar or equatorical, orbits. The results are approximate since we have used a sphere as an approximation for the shape of the Earth. Modifications of these procedures may be developed for determining the latitude and longitude of positions of satellites in other circular orbits about Earth.

2–5 Exercises—Earth—a Satellite with Satellites

1. Consider a satellite with a period of 36 hours that has a circular equatorial orbit and was over the Greenwich meridian at the time of burnout (the time at which the fuel is exhausted). What was the approximate longitude of the satellite 30 hours after burnout?
2. Consider a satellite with a period of 36 hours that has a circular equatorial orbit and was in position over 80° west longitude at the time of burnout. What was the approximate longitude of the satellite 100 hours after burnout?

2–6 Positions of Stars

Have you ever tried to find a particular star? Have you tried to identify or describe the position of a star? Poets consider stars to be "windows of heaven"; physicists consider stars to be sources of energy; astronomers consider stars to be sources of knowledge; and navigators consider stars to be compasses of space.

Many navigators today use their eyesight to locate stars while navigating about Earth. You may ask how this is accomplished during daylight when most stars are not visible. However, remember that the sun is Earth's nearest star and can be used for navigating purposes.

We will discuss a method for locating objects in the sky and a procedure for plotting their position on a chart. The last topic will involve the determination of an equation which best represents these positions on a chart.

Suppose that you were a fire lookout and saw a thin column of smoke on the horizon. How could you describe the position of this fire to other people so that from two or three observations the position could be located affectively? One common way is to describe the direction of the fire with north as a reference direction. For example, observer A in Figure 2–39 might identify the location of the fire as "north 40° east," observer B as "north 20° west," and observer C as "north 85° west." Notice that the observation of C serves as a check on the observations of A and B. The navigator would describe each observation in degrees measured clockwise from north; that is, as 40°, 340°, and 275°. Each of these measures is called the *azimuth* of the direction.

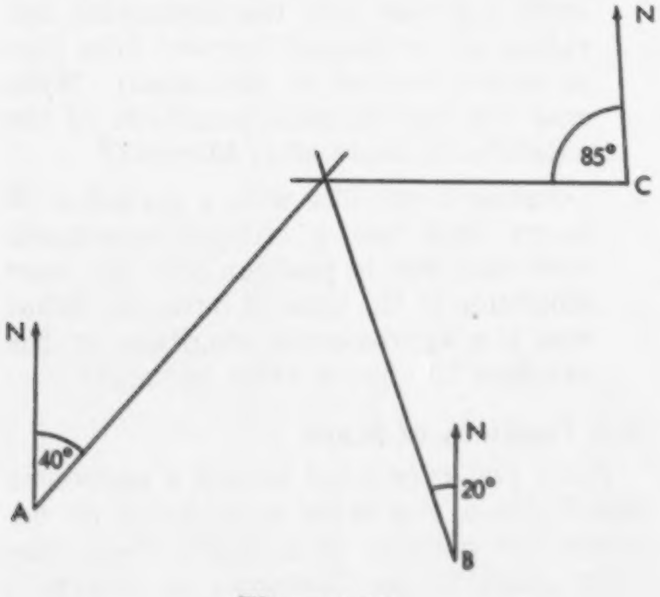


Figure 2–39

When we locate a star we need to know not only the azimuth indicating the horizontal direction in which to look, but also how high in the sky to look. Remember that wherever you are, you have a zenith (Section 2–3) and that your position is on the ray from the center of the Earth and through your zenith. You may think of your zenith as "directly overhead" and your horizon plane as a plane through your position and perpendicular to the ray to your zenith. You may tell a person who is standing beside you how high to look in the sky either using the angle $90° - \theta$ between the ray to the star and the ray to your zenith or using the angle θ between the ray to the star and your horizon (Figure 2–40). This last angle θ is called the *altitude* of the star.

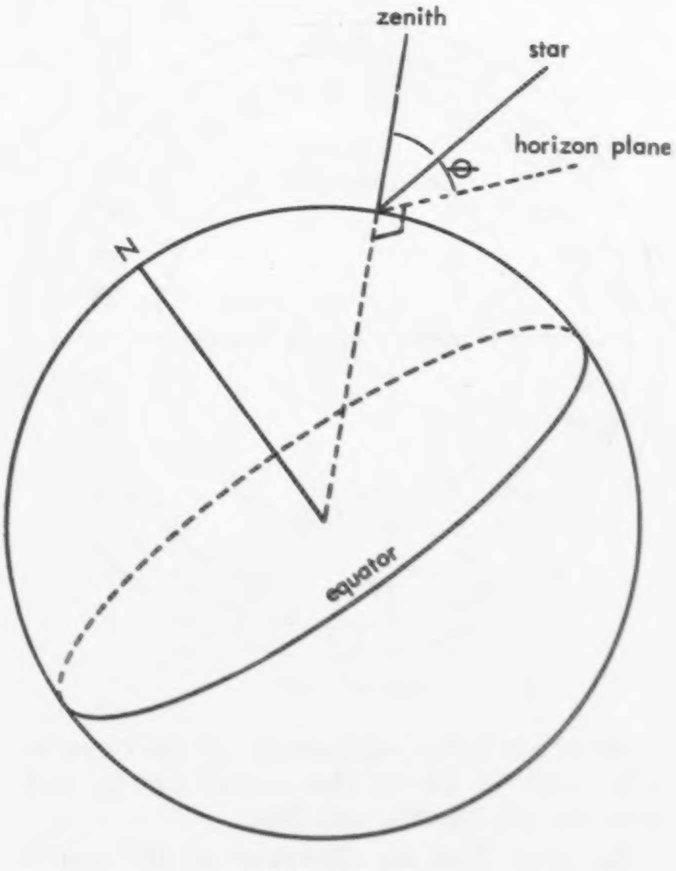


Figure 2–40

We may use the azimuth and the altitude of any star from our position to point to the star. Notice that due to the movement of Earth and the stars, these coordinates are applicable only at a specific time. This "model" description has also been simplified in other minor ways that are studied in more advanced treatments.

As a special case of the location of a star consider the north star, Polaris, the one star whose position remains approximately fixed relative to an observer on Earth. Any observer in the northern hemisphere should be able to find Polaris by looking north (azimuth 0°) and at altitude equal to the latitude of the observer.

Anyone wishing to practice using azimuths and altitudes should try measuring (at a given time) the azimuths and altitudes of stars or other objects such as the moon. It is an interesting experiment to record the azimuth and altitude of the moon over the period of a month at the same time (such as 9 p.m. each night). The data can be plotted on a chart using azimuth for a horizontal axis and altitude for the vertical axis. The resulting chart will show variations in azimuth and altitude over a period of a month. Can you explain why?

2–7 Our Galaxy, the Milky Way

We now turn our attention to the method of determining positions of stars that is used in the study of our galaxy, the Milky Way. Think of the sky above you as part of a huge *celestial sphere* with the center of Earth as its center. The *celestial equator* is the intersection of Earth's equatorial plane with the *celestial* sphere. The *celestial* north and south *poles* are determined by the intersection of Earth's polar axis with the celestial sphere. As in Figure 2–41 the celestial north pole is often designated as C.N.P.

Earth's equator may be divided into 24 parts and numbered indicating hours as the Earth rotates on its axis. The hour marks on the equator appear to be numbered counter-clockwise when viewed from the northern hemisphere, but they appear to be numbered clockwise when viewed from the southern hemisphere.

We now project the hour marks visualized on Earth's equator from Earth's center to the celestial equator. As on Earth's equator the hour marks on the celestial equator determine 24 equal arcs, each arc has arc measure 15°, and the numbering of the hour marks appears counterclockwise when viewed from the celestial north pole.

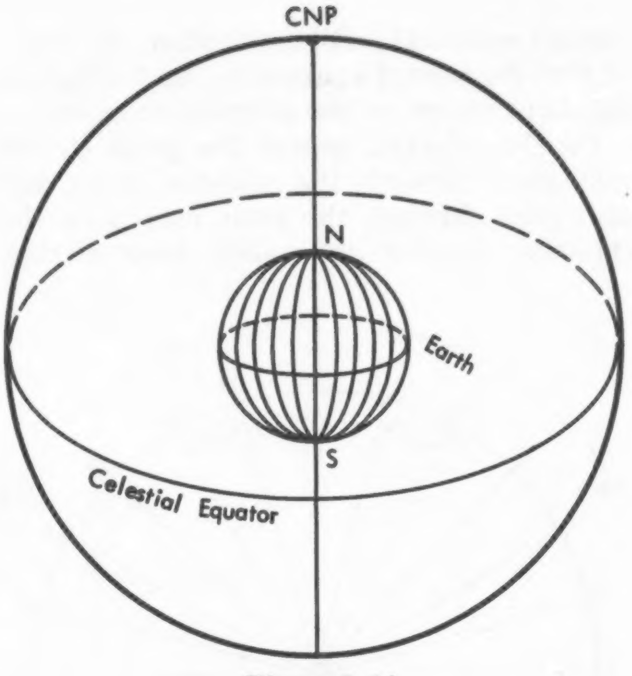
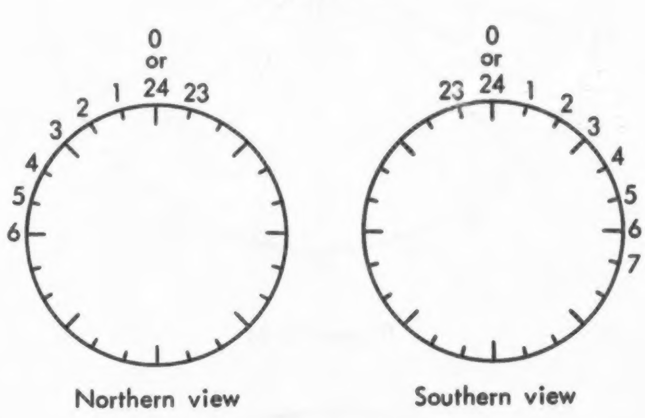


Figure 2–41



Northern view

Southern view

Figure 2–42

Earth appears to rotate counterclockwise when viewed from above its north pole. This means that an observer at Earth's north pole would have to turn clockwise at an angular speed of 15° per hour in order to maintain his orientation relative to the celestial sphere. If the observer stands still on Earth then the rotation of Earth rotates him counterclockwise and the celestial sphere appears to be rotating clockwise.

In order to use the celestial sphere as a reference system we need to fix the positions of the hour marks on the celestial equator. This is done by selecting one of the intersections of the plane of the apparent path of the sun and Earth's (also the celestial) equatorial plane. These intersections occur about March 21 (vernal equinox) and about September 20 (autumnal equinox). By convention the position of the vernal equinox is the 0 (that is, 24) hour mark on the celestial equator.

On the celestial sphere the great circles that pass through the celestial poles and also pass through the hour marks on the celestial equator are called *hour circles.*

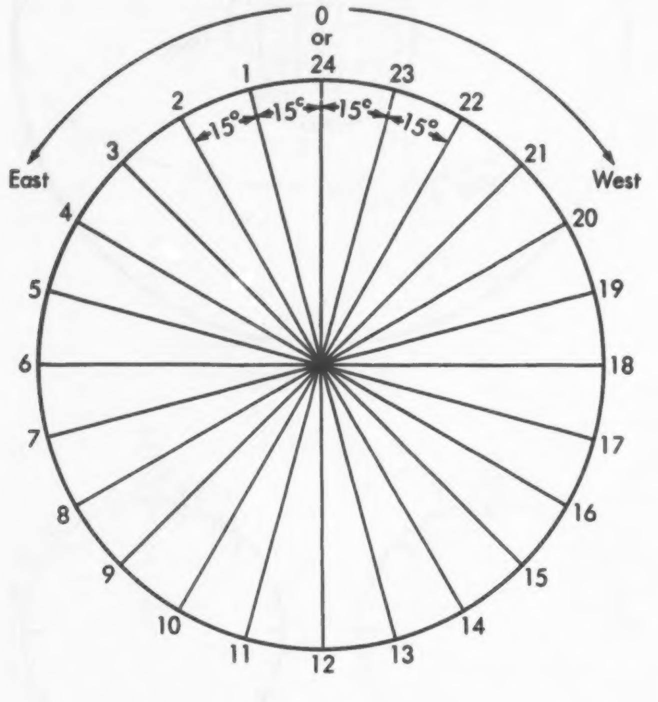


Figure 2-43

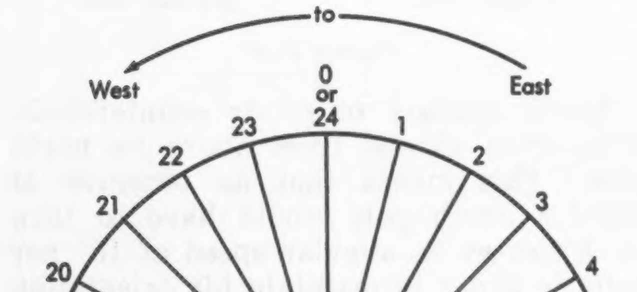


Figure 2-44

The view of these hour circles from the celestial north pole is shown in Figure 2-43. Note the counter clockwise numbering. If these same circles are viewed from Earth's north pole, the view is nearly the same as from the celestial south pole and the numbering of the hour circles appears clockwise as in Figure 2-44.

The hour circles of the celestial sphere are fixed in space and numbered from 1 to 24 counter clockwise from the position of the vernal equinox. Thus each hour is represented by 15° of arc:

1 hour—	15° of arc
4 minutes—	1° of arc
1 minute—	15′ of arc
1 second—	15″ of arc

Any star that appears to be on the third hour circle of the celestial sphere is said to be 3 hours from the position of the vernal equinox. Usually this is abbreviated by saying that the star has a *right ascension* of 3 hours 0 minutes and 0 seconds. The term right ascension is derived from early observations of the rising (ascending) of stars nearly at right angles to the horizon (actually it is only a right angle when the observer is at the equator). However, the right ascension does not indicate how high in the sky one should look for a star. This "height" of a star is specified with reference to the celestial equator.

For any observer the celestial equator is on a plane through the position of the observer and perpendicular to his line of sight to the celestial north pole. The line of sight to celestial north pole is approximately (within 1° of) the line of sight to the north star (Polaris). The celestial north pole is at the zenith of an observer at Earth's north pole. For any other observer in the northern hemisphere the celestial north pole has azimuth 0° and altitude equal to the observer's latitude. Observers in the southern hemisphere may find the celestial south pole.

We may now identify the position of any star by the intersection with the celestial sphere of our line of sight to the star. This intersection will be designated by its right ascension and its declination (+ if the altitude is measured above the celestial equator,— if below). By convention the right

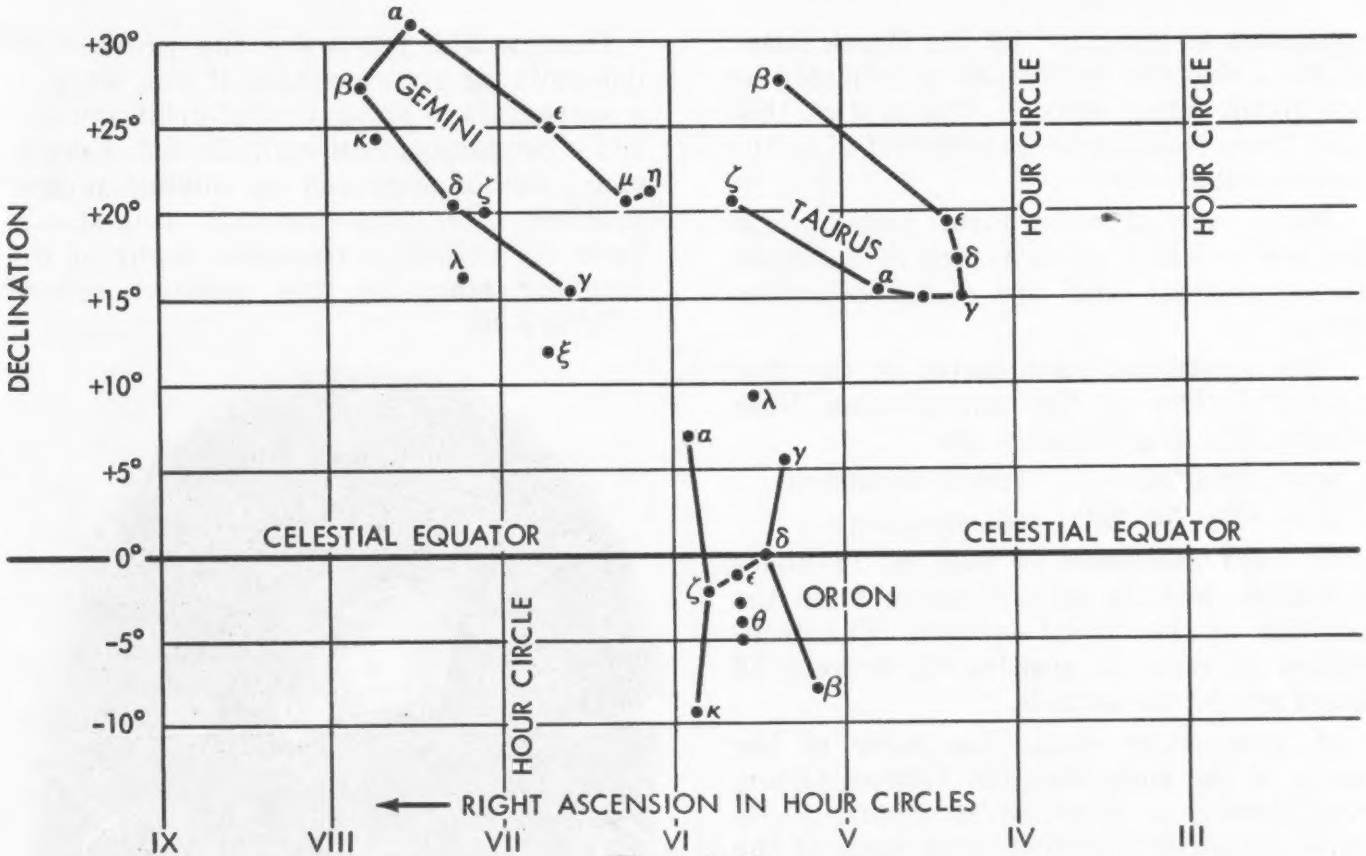

DECLINATION
+30°
+25°
+20°
+15°
+10°
+5°
0°
-5°
-10°
α
β
κ
GEMINI
ε
μ
η
δ
ζ
λ
γ
ξ
β
ζ
TAURUS
ε
δ
α
γ
λ
α
γ
δ
ε
ζ
θ
ORION
β
κ
CELESTIAL EQUATOR
CELESTIAL EQUATOR
HOUR CIRCLE
HOUR CIRCLE
HOUR CIRCLE
RIGHT ASCENSION IN HOUR CIRCLES
IX
VIII
VII
VI
V
IV
III

Figure 2–45

TABLE 2–7

Equatorial Coördinates of Stars in Figure 2–45

Star and Constellation	Name	(To nearest minute of time) Ascension Right	(To nearest minute of arc) Declination
γ Taurus	—	4^h18^m	+15°33′
δ Taurus	—	4^h21^m	+17°28′
ε Taurus	—	4^h27^m	+19°06′
α Taurus	Aldebaran	4^h34^m	+16°26′
β Taurus	—	5^h24^m	+28°35′
ζ Taurus	—		
β Orion	Rigel	5^h13^m	− 8°14′
γ Orion	Belatrix		
δ Orion	—		
ε Orion	—		
ζ Orion	—		
κ Orion	—		
α Orion	Betelgeuse	5^h53^m	+ 7°24′
η Gemini	—		
γ Gemini	—		
α Gemini	Castor	7^h32^m	+31°58′
β Gemini	Pollux		

ascension is indicated by the Greek letter alpha α and the declination is indicated by the Greek letter delta δ. The system that uses these coordinates is referred to as the *equatorial system.*

Many sets of stars form patterns (in the sky and also when plotted in the equatorial system) and are called *constellations.*

The equatorial coordinates of the star named Dubbe in the constellation Ursa Major (the Big Dipper) are:

$\alpha = 11^h01^m35^s$ (right ascension)
$\delta = +61^\circ\, 56'\, 25''$ (declination)

The right ascension is read as 11 hours 1 minute and 35 seconds away from the position of the vernal equinox. The declination is read as positive 61 degrees 56 minutes and 25 seconds.

A star chart containing some of the stars of the constellations Taurus, Orion, and Gemini is sketched in Figure 2–45. The equatorial coordinates of some of the stars contained in these constellations are listed in Table 2–7. Only the prominent stars have been named in the table. Some of the equatorial coordinates have been omitted from the table; these can be estimated from the star chart in Figure 2–45.

Man seems to be taking his first feeble steps away from Earth and out into space. This new venture and exploration of a new frontier involves many problems which must be solved correctly if man is to succeed. We will consider only the problem of locating one's position in space.

Magnetic compasses cannot be relied upon. The compass to be considered here is a clock and several known stars contained at fixed positions relative to the Earth.

Consider the problem of an astronaut attempting to fix his position without radio communication with Earth. Our hypothetical problem will be limited to an equatorial orbit of eccentricity zero. This means that the orbital path is circular about Earth and is contained in the equatorial plane.

The manned space craft Gemini—Titan 4, GT—4, was in circular orbit but the inclination of orbit was approximately 29°.

How would you solve the problem of determining your position if you were in a spacecraft in an equatorial orbit and lost radio communication with Earth? Here is one possible approach to solving such a problem. For this approach you should have the ability to recognize many of the brighter stars on the celestial sphere (Table 2–8).

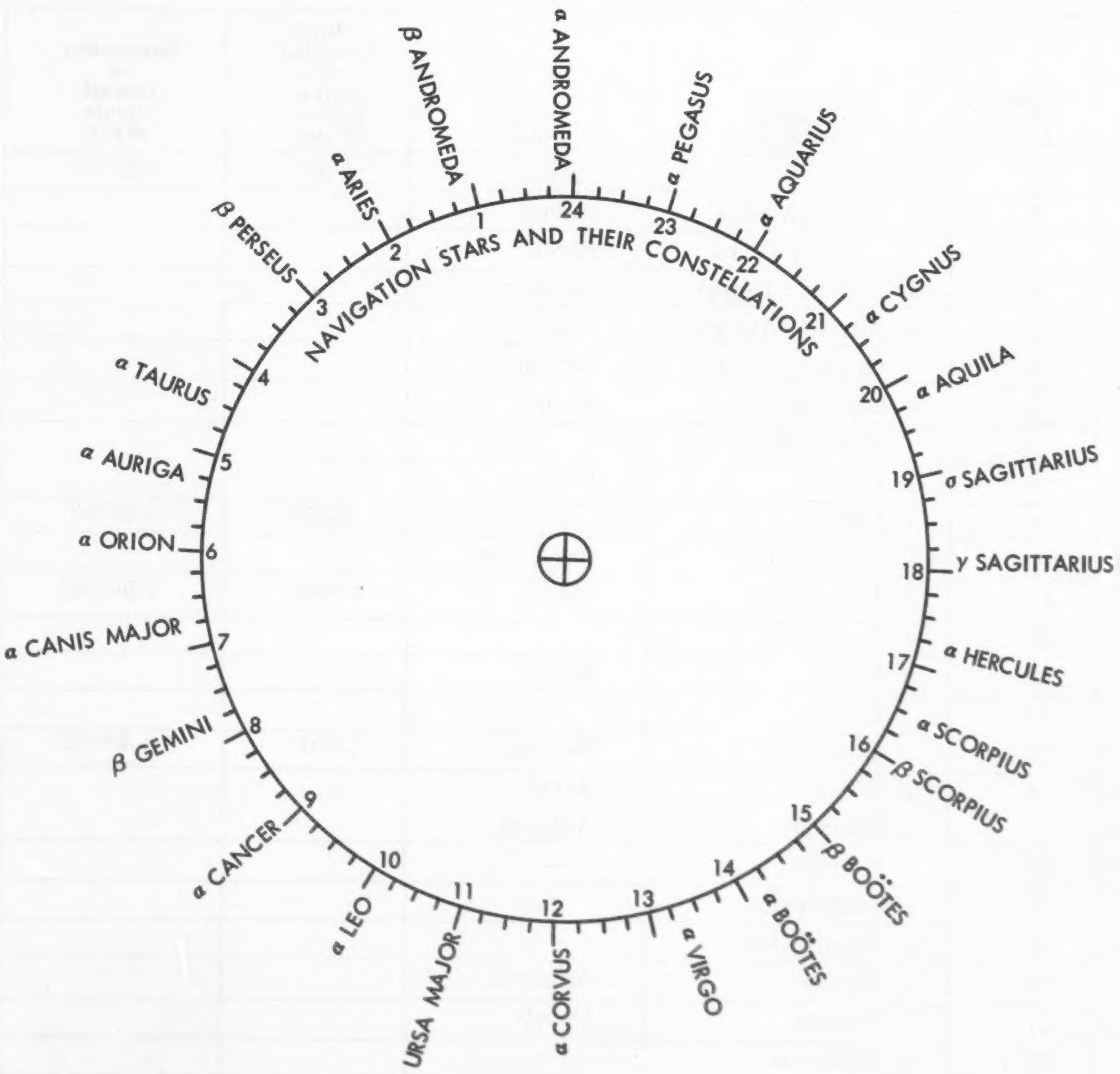
Figure 2–46

The 25 stars listed in Table 2–8 were selected because of their position on the celestial sphere in terms of right ascension and apparent brightness. An attempt was made to select stars near the celestial equator but this was not always possible. The equatorial orbit makes it desirable to use stars having a declination d such that $-25^\circ < d < 25^\circ$. The main criteria used to select these 25 stars was their brightness and nearness to an hour circle on the celestial sphere beginning with the first hour circle.

Figure 2–47 shows the positions of the stars listed in Table 2–8 with reference to hour circles (northern hemisphere view).

TABLE 2–8

Nearest Hour Circle	Letter Position Constellation	Star Name	Right Ascension to nearest minute of time	Declination to nearest minute of arc
1	β Andromeda	Mirach	1^h08^m	$+35°26'$
2	α Aries (Arietis)	Hamal		
3	β Perseus (Persei)	Algol		
4	α Taurus (Tauri)	Aldebaran		
5	α Auriga (Aurigoe)	Capella		
6	α Orion (Orionis)	Betelgeuse		
7	α Canis Major	Sirius		
	(Canis Majoris)			
8	β Gemini	Pollux		
9	α Cancer	—	8^h57^m	$+12°00'$
10	α Leo (Leonis)	Cor Leonis		
11	β Ursa Major	Merak	11^h00^m	$+56°34'$
12	α Corvus	—		
13	α Virgo	Spica		
14	α Boötes	—		
15	β Boötes	Nekkar	15^h01^m	$+40°32'$
16	β Scorpius α Scorpius	Acrab Antares		
17	α Hercules	—		
18	γ Saggitarius	—		
19	σ Saggitarius	—		
20	α Aquila	Altair		
21	α Cygnus	Deneb		
22	α Aquarius	—		
23	α Pegasus	Markab	23^h03^m	$+15°01'$
24	α Andromeda	Alpheratz		


NAVIGATION STARS AND THEIR CONSTELLATIONS
1
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19
20
21
22
23
24
α ANDROMEDA
β ANDROMEDA
α ARIES
β PERSEUS
α TAURUS
α AURIGA
α ORION
α CANIS MAJOR
β GEMINI
α CANCER
α LEO
β URSA MAJOR
α CORVUS
α VIRGO
α BOÖTES
β BOÖTES
β SCORPIUS
α SCORPIUS
α HERCULES
γ SAGITTARIUS
σ SAGITTARIUS
α AQUILA
α CYGNUS
α AQUARIUS
α PEGASUS

Figure 2-47

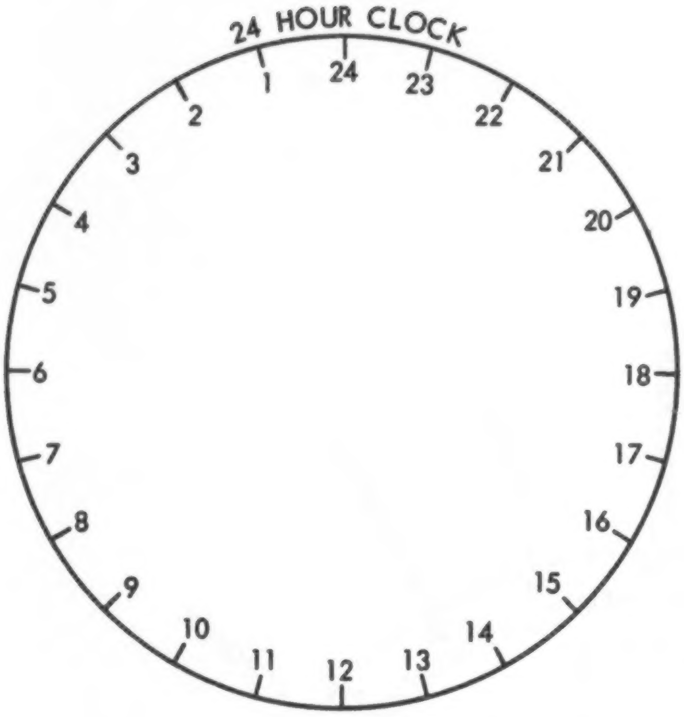

Figure 2–48

Figure 2–48 shows a 24 hour clock numbered counterclockwise.

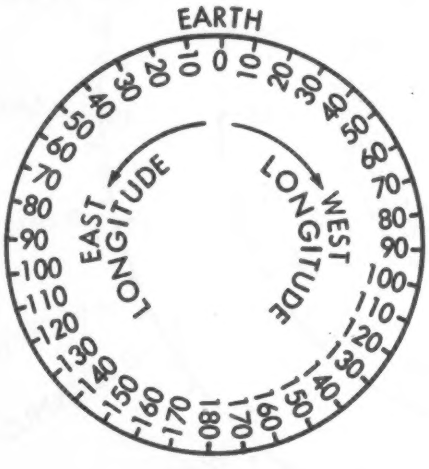

Figure 2–49

Figure 2–49 shows Earths' equator marked in 10° intervals. Figures 2–47, 2–48, and 2–49 may be constructed and mounted with the same centers as in Figure 2–50 page 52. The inner circle should be mounted so that it can be rotated.

The meridian over which the satellite is located at the time of "burnout" is used to position the inner circle in Figure 2–50. The longitude of this meridian is aligned with the 24 hour marks on the other two circles as shown in Figure 2–51. At the time of burnout a watch should be set at zero (that is, 24) hours.

Consider the problem of finding the longitude of the meridian over which a spacecraft is located 66 hours after burnout if burnout occurred over 80° west longitude and the period of the oribit is not known. Suppose also that 66 hours after burnout the satellite, Earth, and α Andromeda are approximately on a line. Since

$$0 + 66 \equiv 18 \pmod{24},$$

we align 80° west longitude on the inner circle of Figure 2–50 with the 18 hour marks on the other two circles as in Figure 2–52.

We next draw a ray from the center of Figure 2–52 page 54 to α Andromeda. This crosses the inner circle at the longitude over which the satellite is positioned; in this case about 12° east longitude.

This position may be checked if the period of orbit of the satellite is known. Suppose that the period in our example was 1.5 hours. Then the satellite would make 44 complete revolutions in 66 hours and should be positioned over the meridian at 10° east longitude as indicated by the ray to the 24 hour marks in Figure 2–53. In this figure an additional circular scale has been added to show the fractional parts of completed orbits. When this scale is used the ray is considered as drawn through 1.5 (the 0 of that scale) since

$$0 + 66 = 0 \pmod{1.5}$$

Notice the error of 2° in longitude (shaded) missing from slight inaccuracies in the observations (such as the alignment of the spacecraft, α Andromeda and Earth) or the calculations (such as the period).

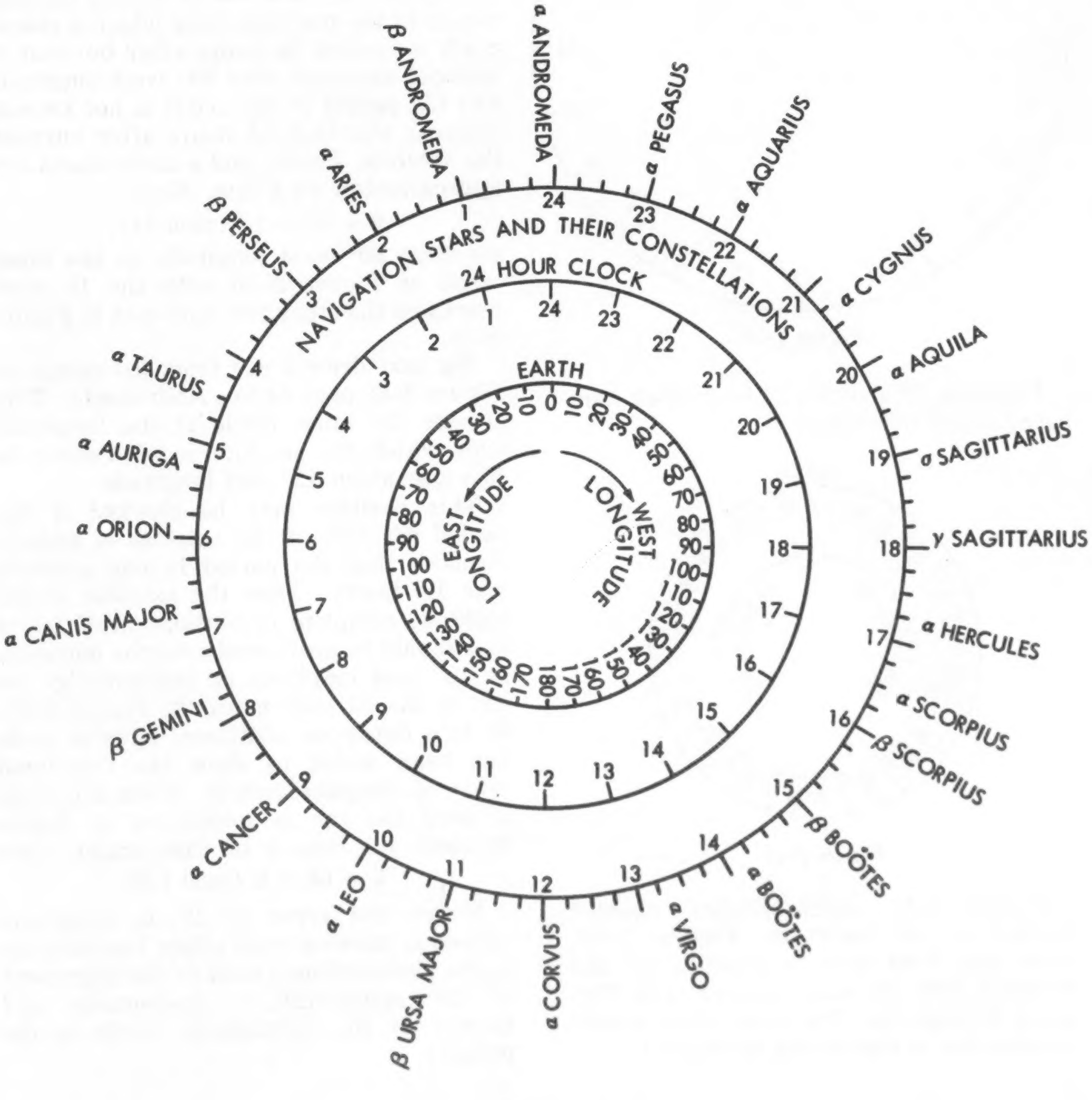
NAVIGATION STARS AND THEIR CONSTELLATIONS
24 HOUR CLOCK
EARTH
EAST LONGITUDE
WEST LONGITUDE
α ANDROMEDA
α PEGASUS
α AQUARIUS
α CYGNUS
α AQUILA
σ SAGITTARIUS
γ SAGITTARIUS
α HERCULES
α SCORPIUS
β SCORPIUS
β BOÖTES
α BOÖTES
α VIRGO
α CORVUS
β URSA MAJOR
α LEO
α CANCER
β GEMINI
α CANIS MAJOR
α ORION
α AURIGA
α TAURUS
β PERSEUS
α ARIES
β ANDROMEDA

Figure 2–50

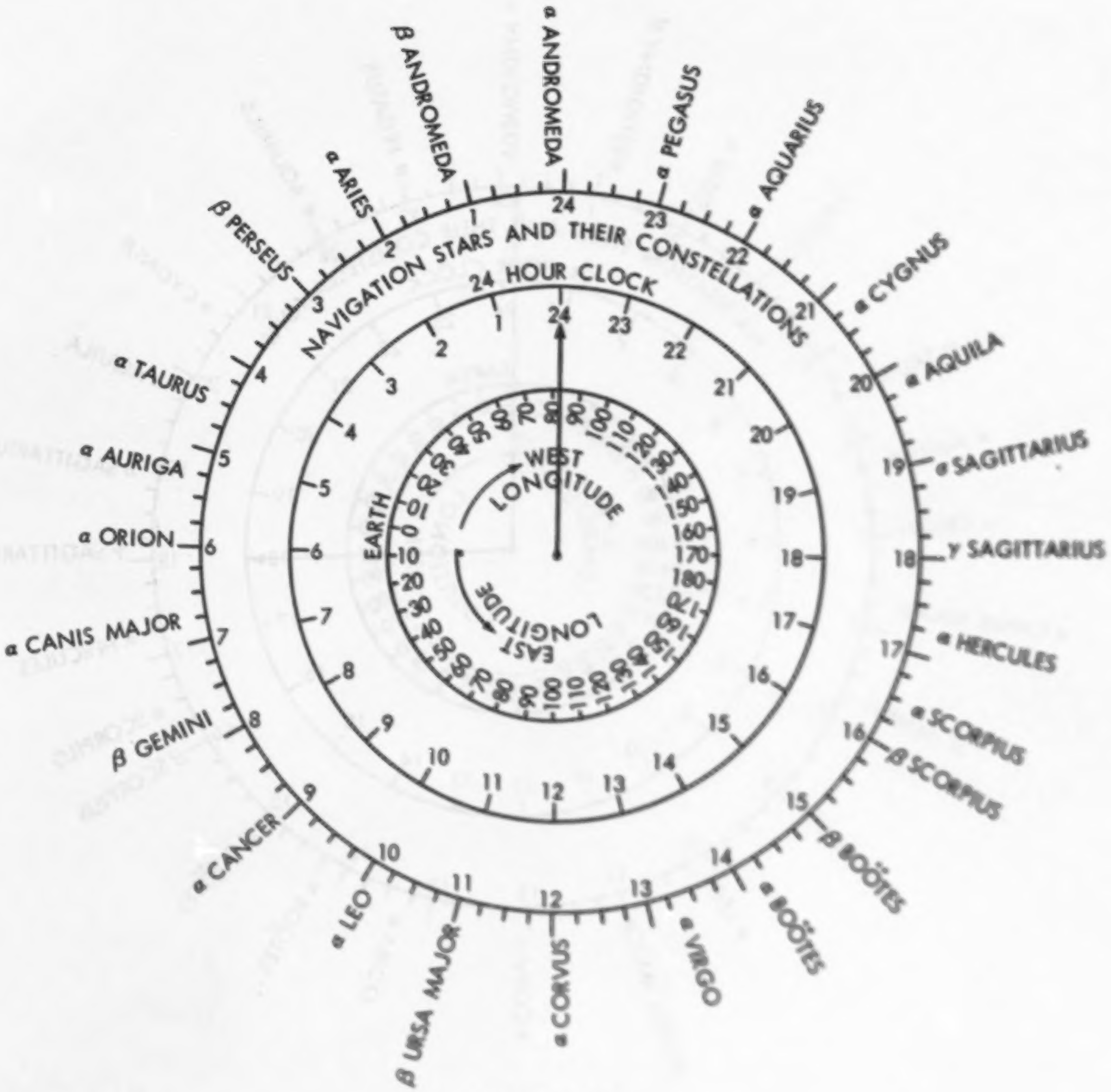
NAVIGATION STARS AND THEIR CONSTELLATIONS
24 HOUR CLOCK
EARTH
WEST LONGITUDE
EAST LONGITUDE
α ANDROMEDA
α PEGASUS
α AQUARIUS
α CYGNUS
α AQUILA
α SAGITTARIUS
γ SAGITTARIUS
α HERCULES
α SCORPIUS
β SCORPIUS
β BOÖTES
α BOÖTES
α VIRGO
α CORVUS
β URSA MAJOR
α LEO
α CANCER
β GEMINI
α CANIS MAJOR
α ORION
α AURIGA
α TAURUS
β PERSEUS
α ARIES
β ANDROMEDA
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Figure 2–51

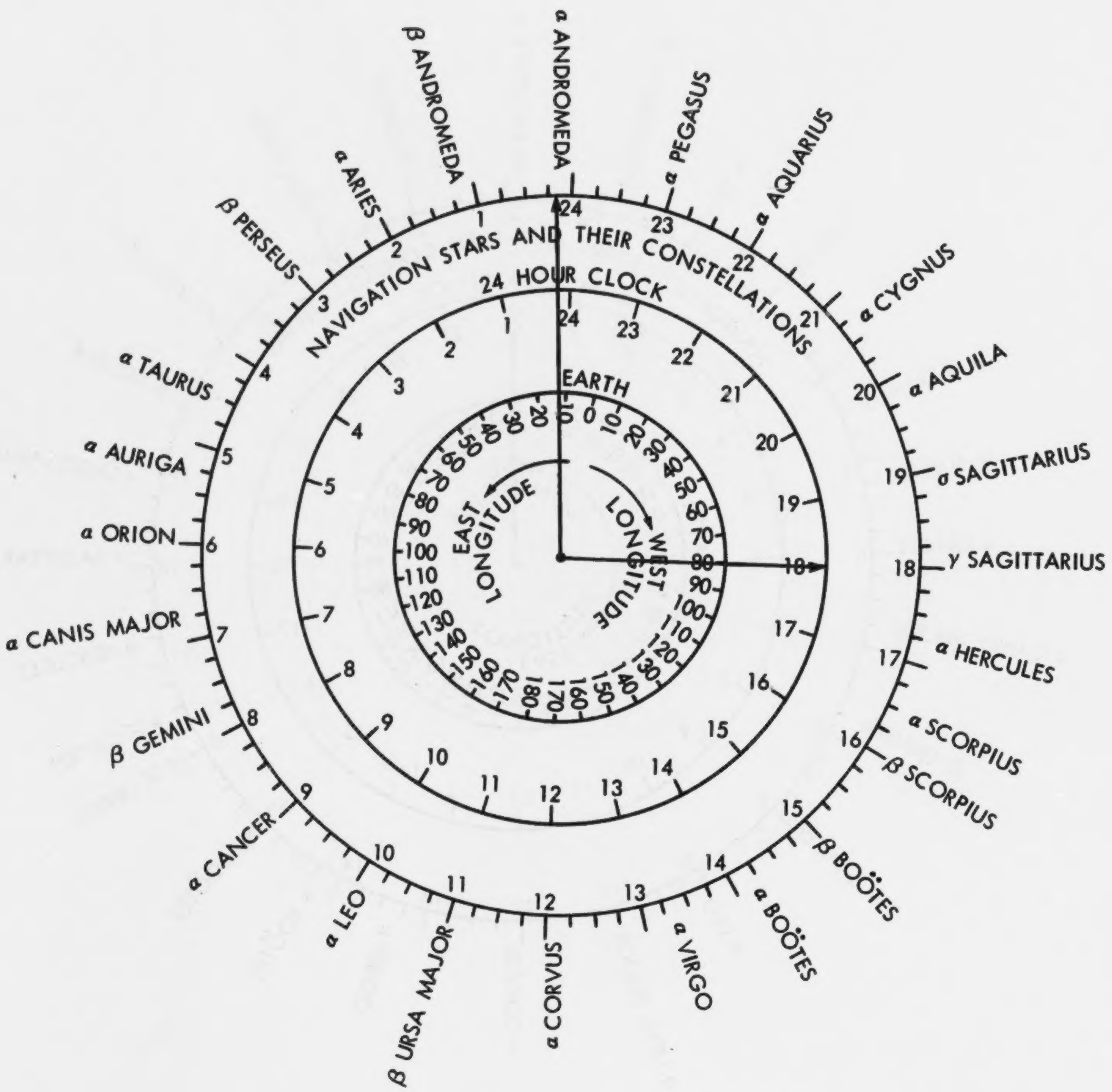
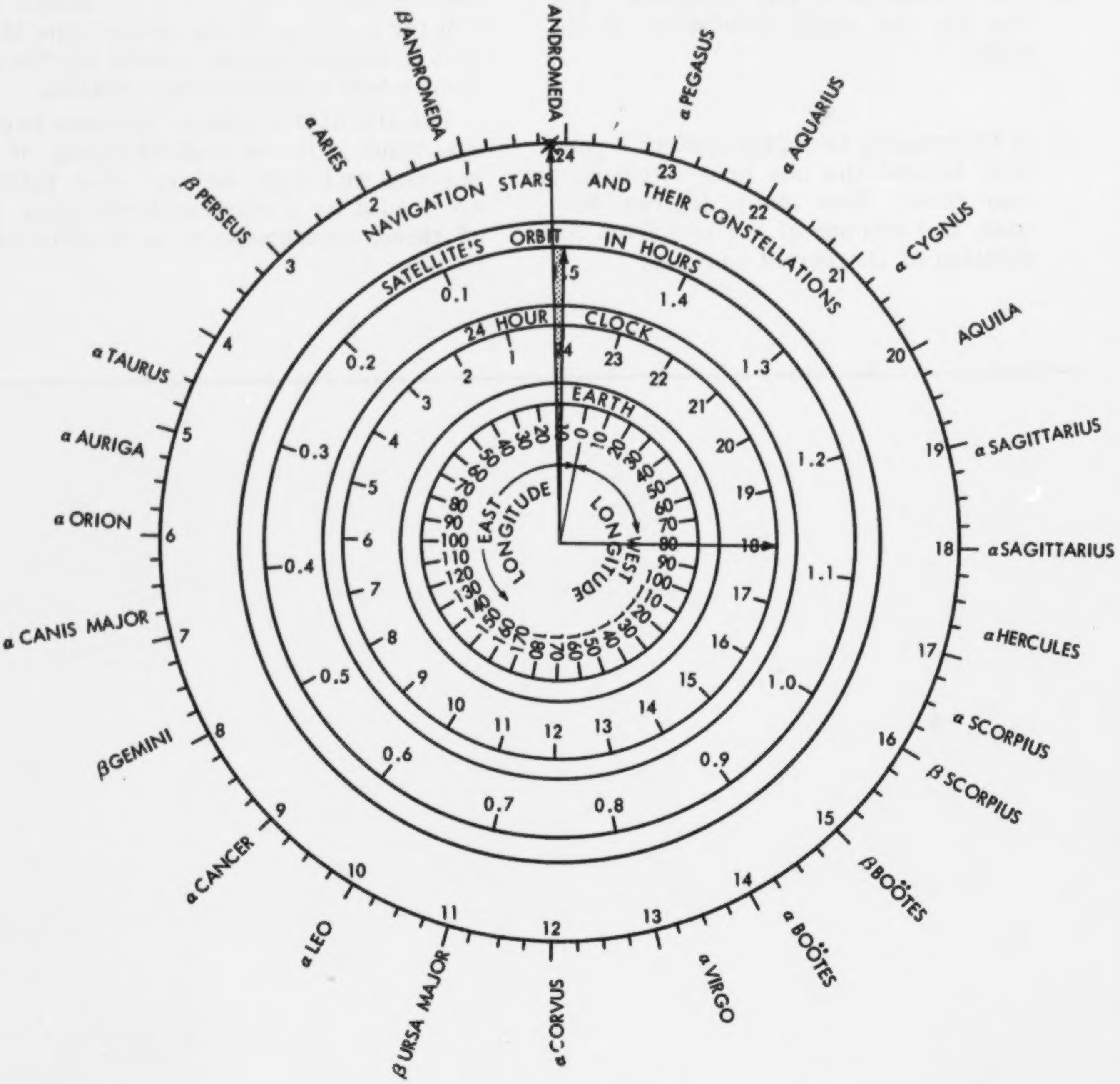

NAVIGATION STARS AND THEIR CONSTELLATIONS
24 HOUR CLOCK
EARTH
EAST LONGITUDE
WEST LONGITUDE
α ANDROMEDA
α PEGASUS
α AQUARIUS
α CYGNUS
α AQUILA
σ SAGITTARIUS
γ SAGITTARIUS
α HERCULES
α SCORPIUS
β SCORPIUS
β BOÖTES
α BOÖTES
α VIRGO
α CORVUS
β URSA MAJOR
α LEO
α CANCER
β GEMINI
α CANIS MAJOR
α ORION
α AURIGA
α TAURUS
β PERSEUS
α ARIES
β ANDROMEDA

Figure 2–52

2


NAVIGATION STARS AND THEIR CONSTELLATIONS
SATELLITE'S ORBIT IN HOURS
24 HOUR CLOCK
EARTH
EAST LONGITUDE
WEST LONGITUDE
α ANDROMEDA
β ANDROMEDA
α ARIES
β PERSEUS
α TAURUS
α AURIGA
α ORION
α CANIS MAJOR
βGEMINI
α CANCER
α LEO
β URSA MAJOR
α CORVUS
α VIRGO
α BOÖTES
β BOÖTES
β SCORPIUS
α SCORPIUS
α HERCULES
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AQUILA
α CYGNUS
α AQUARIUS
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0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4

Figure 2–53

2-7 Exercise—Our Galaxy the Milky Way

1. Use the star chart in Figure 2-45 to complete Table 2-7 for the right ascensions and declinations to the nearest minute of time and arc.

2. Use Figure 2-50 and complete Table 2-8 for the right ascensions of the stars.

3. β Andromeda is approximately 8 minutes beyond the one hour circle on a star chart. How many degrees, minutes, and seconds of arc is β from the position of the vernal equinox?

2-8 The Universe

The discussion of coordinate systems in this chapter provides only an introduction to the systems used in mathematics and the many branches of science. For example, we have used right ascension and declination to identify the direction to a star. In Chapter 3 the distances to stars are considered. There is a coordinate system for positions in our galaxy (the Milky Way); another system is used for the universe which contains many galaxies.

The aim of this chapter has been to open one window to an understanding of the universe we live in. Several other windows are needed for a sound understanding. One of these, measurement, is considered in Chapter 3.

Chapter 3

MEASUREMENT
A WINDOW TO THE UNIVERSE

by
John Soroka
Planetarium Director
Akron Public Schools
Akron, Ohio

MEASUREMENT, A WINDOW TO THE UNIVERSE

We can see so many stars in the night sky we are unable to count them. It is not surprising that early man thought of himself as standing in the center of a huge sphere with the stars fixed on it like thousands of shining points.

He was not content however with the visible appearance of these celestial bodies and set out to investigate them. As he tried to express their measures and to place each in its proper position, he concluded that the sun is much larger than Earth; that Earth travels around *(orbits)* the sun. As Earth orbits the sun, the moon orbits Earth at a distance almost insignificant in relation to Earth's distance from the sun.

We are part of a very small planetary system in an enormous universe, the measure of which can be dealt with in much the same manner as a surveyor measures the width of a river he cannot cross. We employ these methods to determine the positions, distances, sizes, and motions of the nearer celestial bodies. Knowledge of celestial bodies not only has enabled us to stretch out into space but very shortly will enable a man to step out on the surface of the moon.

3–1 Direct Linear Measurement

Measurement enables us to relate science and mathematics. The accuracy of a measurement can determine the accuracy of a scientific investigation.

We do not always "see" what we are looking at. Consider some of the relationships among the objects shown in Figure 3–1.

The grapefruit is larger than the orange.

The length of the driveway is larger than its width.

The doorway is higher than it is wide.

You probably feel certain that these statements are correct.

Look at the picture in Figure 3–2. Is the chair nearer the door or the window? In order to answer such a question, we might measure the actual distances in

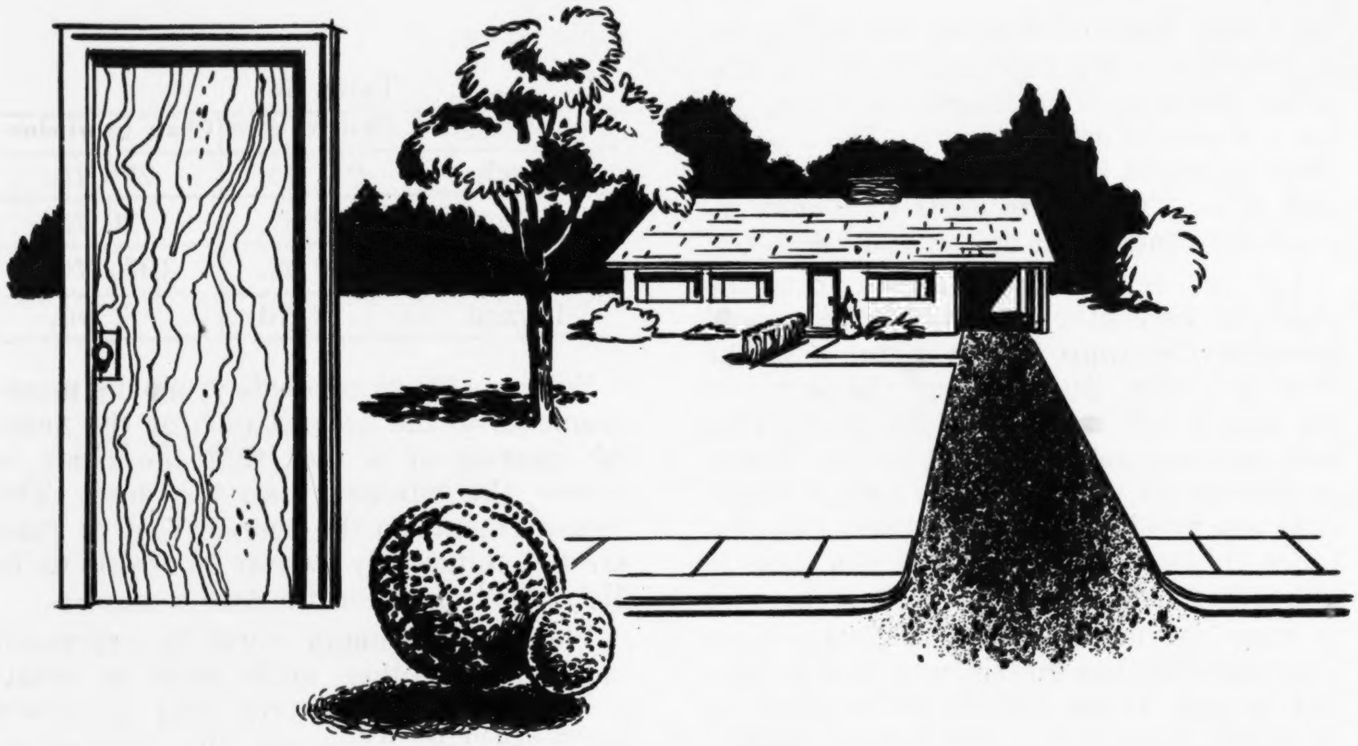


Figure 3–1

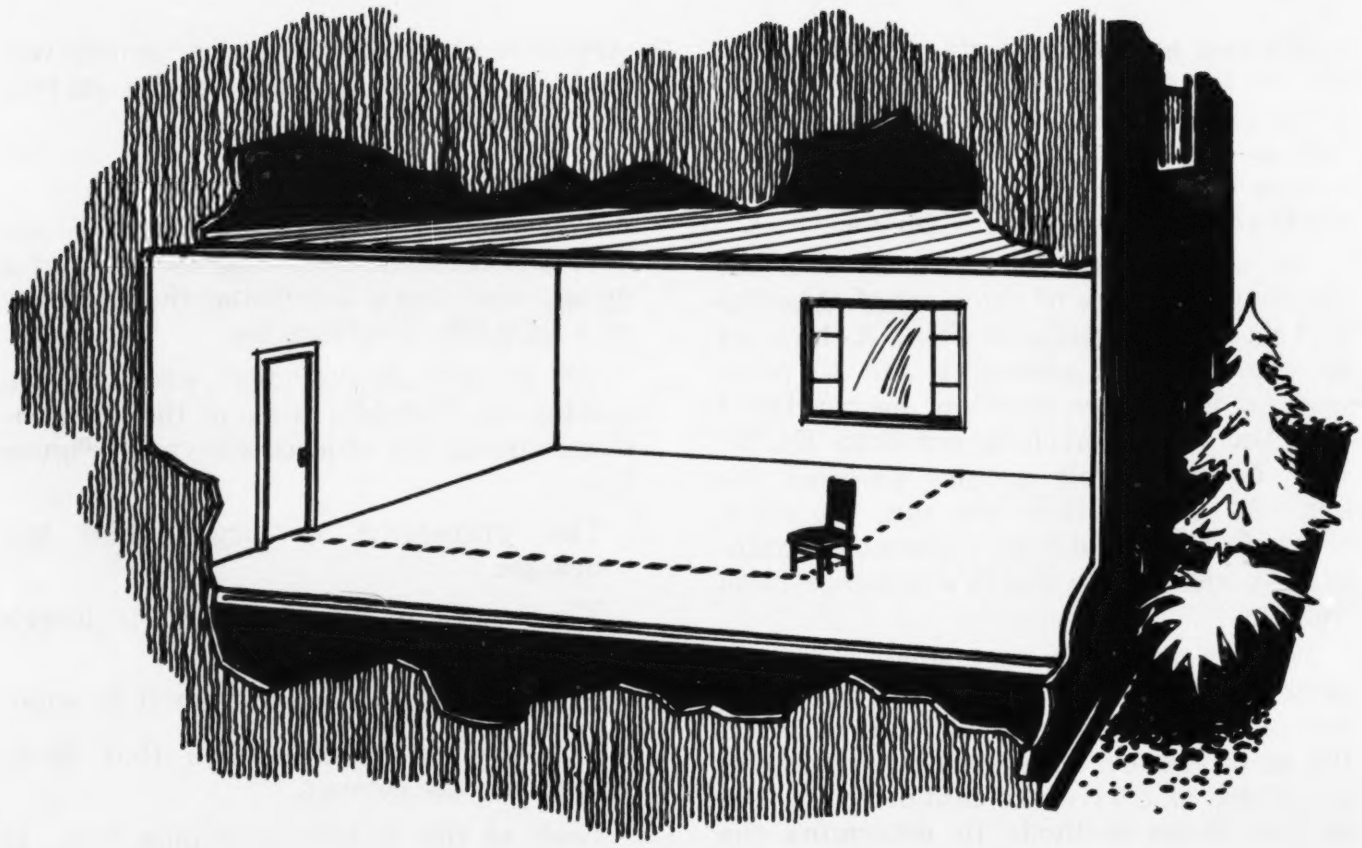


Figure 3–2

the room. The unit that we choose is purely arbitrary. We might pace off the distance, measure it in lengths of string, or use a standard unit of length. The ***English units of length*** include the inch, foot, yard, and mile. The ***metric units of length*** include the centimeter, meter, and kilometer.

Let us consider the distance from the chair to the center of the door. We could measure the approximate distance to the door in inches. Suppose that the center of the door is 241 inches from the chair. This may be true, yet a distance of 241 inches is difficult to visualize. We cannot "see" it in our minds. If we remeasure this distance in feet, we'll find that the door is about 20 feet from the chair. This distance is more meaningful to most people since they can visualize 20 feet more easily than 241 inches. If we remeasure the distance in yards, we find that the door is about 7 yards from the chair. The results of several measurements are summarized in the table below.

Table 3–1.

Unit of length	Chair to door	Chair to window
1 inch	241 in.	236 in.
1 foot	20 ft.	20 ft.
¼ foot	20 ft.	19¾ ft.
1 yard	7 yd.	7 yd.

We are able to conclude from the measurements to the nearest inch or the nearest quarter of a foot that the chair is nearer the window than the door. The measurements to the nearest foot or yard are not sufficiently *precise* to enable us to distinguish between the two distances.

The measurements could be expressed in terms of other units such as centimeters (cm.) and meters (m.) as shown in Table 3–2. However, the conclusions reached do not depend upon the system of units used. The unit is a convenience.

TABLE 3-2

Unit of length	Chair to door	Chair to window
1 centimeter	612 cm.	599 cm.
1 meter	6 m.	6 m.

There is a great temptation to convert results from English units to metric or from metric units to English. In this chapter we will not convert back and forth; but will think and work in appropriate units. Because the measure of spaces requires units which we cannot experience or "see," let us begin with some familiar ones. We can "see" that the distance represented by 1 inch is larger than the distance represented by 1 centimeter and that 1 meter represents a greater distance than 1 yard.

The length of the line segment in Figure 3–3 may be measured to the nearest half inch and to the nearest centimeter as:

3½ inches, or 9 centimeters.

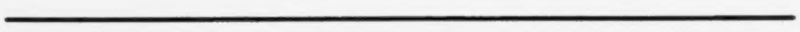


Figure 3–3

We find not a whole number, but a certain number plus a fraction. We will try to estimate the fraction of a centimeter or inch. Inches and centimeters sometimes are divided into ten equal parts. Each space represents one-tenth of the unit. We are able to estimate the length of the line segment in Figure 3–3 to the nearest tenth in each case as :

3.4 inches.
8.7 centimeters.

We may observe that the line segment is not exactly 3.4 inches. The length of the line segment is between 3.4 and 3.5 inches. If we were not concerned with greater accuracy, we could see that the length of the line segment is closer to 3.4 than 3.5 inches and simply say 3.4 inches is the length. We may imagine the tenths divided into ten equal parts and then estimate the length of the line segment to the nearest hundredth of an inch and to the nearest hundredth of a centimeter. You can see that we can never describe the measure of the line segment exactly.

If you and a friend both estimate the length of the line segment in Figure 1–3, you probably will agree on the number of inches and the number of tenths of an inch. You may not agree on the number of hundredths of an inch. If not, try again. Who is right? The answer to the question will depend upon your abilities to estimate distances.

3–1 Exercises — Direct Linear Measurement

1. Use a yardstick and a meter stick to compare the length and width of a room. Compare the ratios of the lengths to the widths in each of the two systems of units. Are they the same?
2. Make (or have ten friends make) ten measurements of the line segment in Figure 3–3. Add the numbers obtained and divide by ten. How does this *average* value compare with the others? Does this average value appear more "reliable" than the individual values?

3–2 Direct Angular Measurement

Angular measurement may also be used in determining distances. Consider an angle as the union of two rays with a common end point as in Figure 3–4.

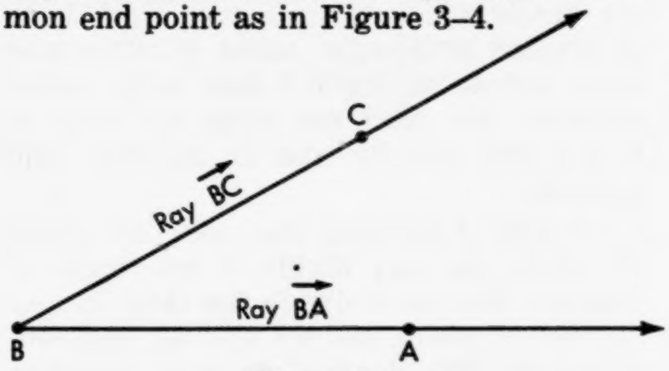
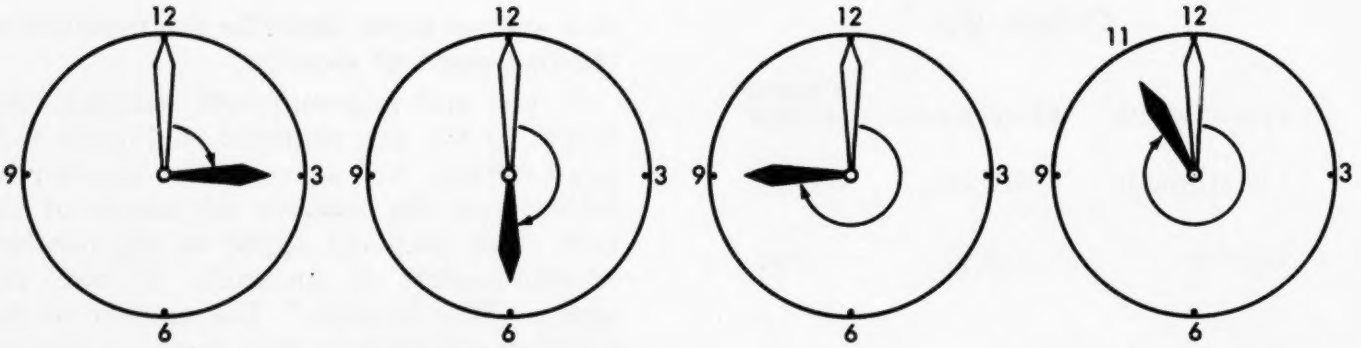


Figure 3–4

In measuring an angle, we need a unit of angular measure. Remember the selection of a unit of measurement is arbitrary. The angular unit of measurement for figures on a clock is the hour (Figure 3–5); for the circle it is usually the degree (Figure 3–6).



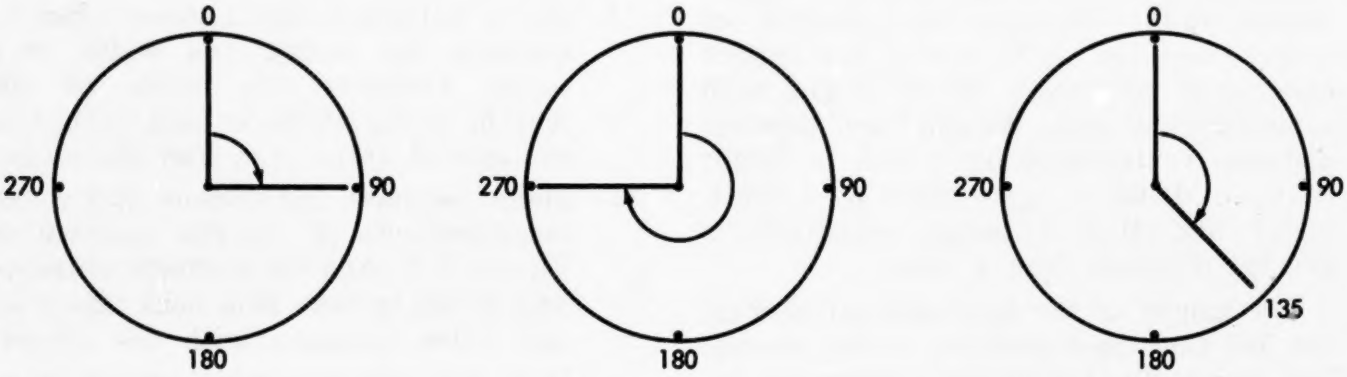
1/4 of a circle 3 hours of time

1/2 of a circle 6 hours of time

3/4 of a circle 9 hours of time

11/12 of a circle 11 hours of time

Figure 3–5



1/4 of a circle 90°

3/4 of a circle 270°

3/8 of a circle 135°

Figure 3–6

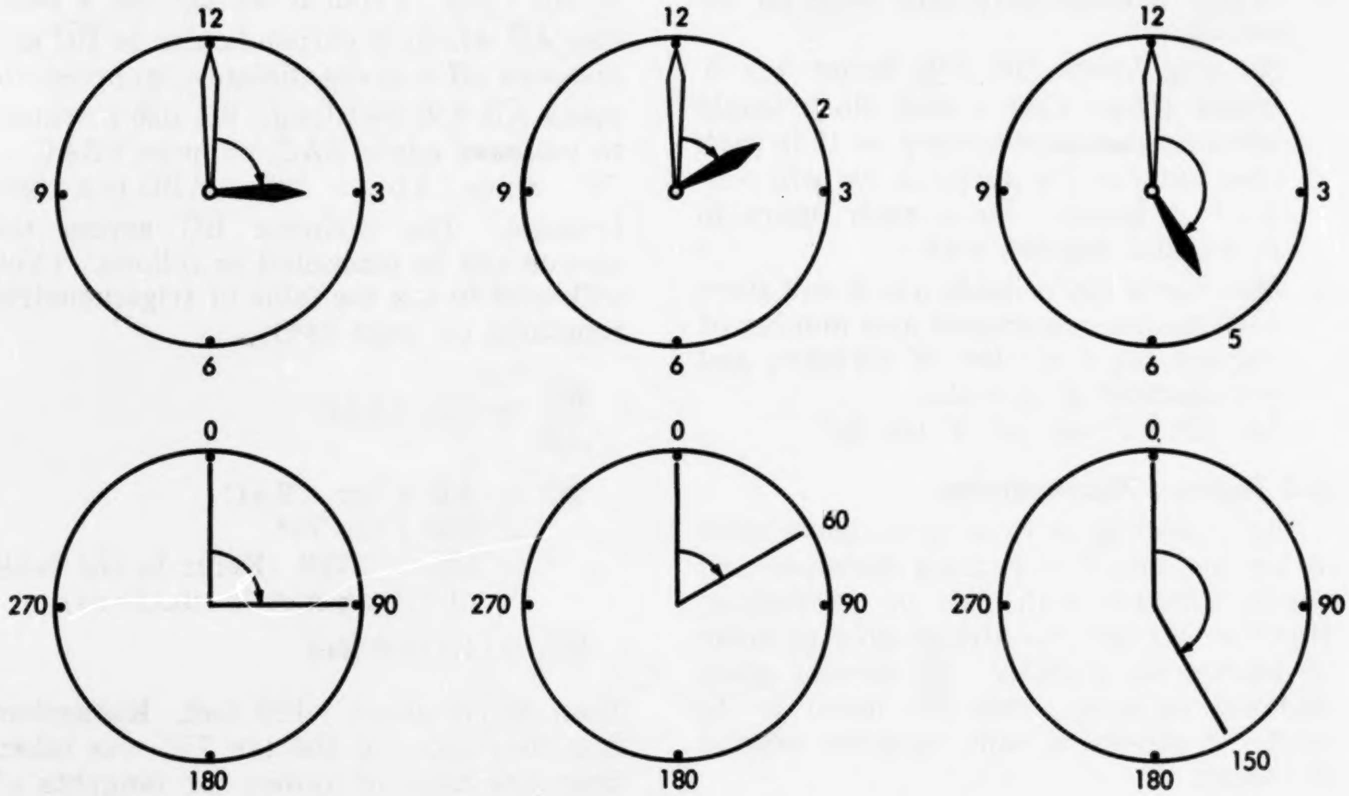
The circles representing the clocks are divided into 12 equal parts called hours. When we read the time from the clock, we are reading the angle represented by the hands which form a ray. Each hour is divided into units called minutes and each minute is divided into units called seconds. We read our time not only in hours, but usually also in minutes and seconds.

Instead of dividing the clock into units of hours, we may divide it into units of degrees. We could divide the clock in any number of units, but we will use the conventional 360 degrees in one complete revolution (circle).

We could measure the angle formed by the hands of a clock either in hours or in degrees.

Figure 3–7 enables us to compare the sizes of the angles represented in hours or in degrees. The angle represented by three hours equals the angle represented by 90 degrees and so on.

Figure 3–8 enables us to read other angles in hours or degrees. The angle represented in part (a) would be 9 hours; that is, 270 degrees. The angle represented in part (b) would be 3 hours; that is, 90 degrees. The angle represented in part (c) would be 1 hour and 30 minutes; that is, 45 degrees. Again we have the temptation to convert from one unit to the other but should continue to think in any convenient unit. In the remainder of this chapter, we will speak of angles in degrees. The symbol used to indicate the word degree is °. If we need greater accuracy, we can divide the degree into smaller parts called minutes; one degree equals 60 minutes. The minute is divided into units called seconds; one minute equals 60 seconds. These are *not* units of time but parts of a degree. The terms


12
9
3
6
12
2
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3
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12
9
3
5
6
0
270
90
180
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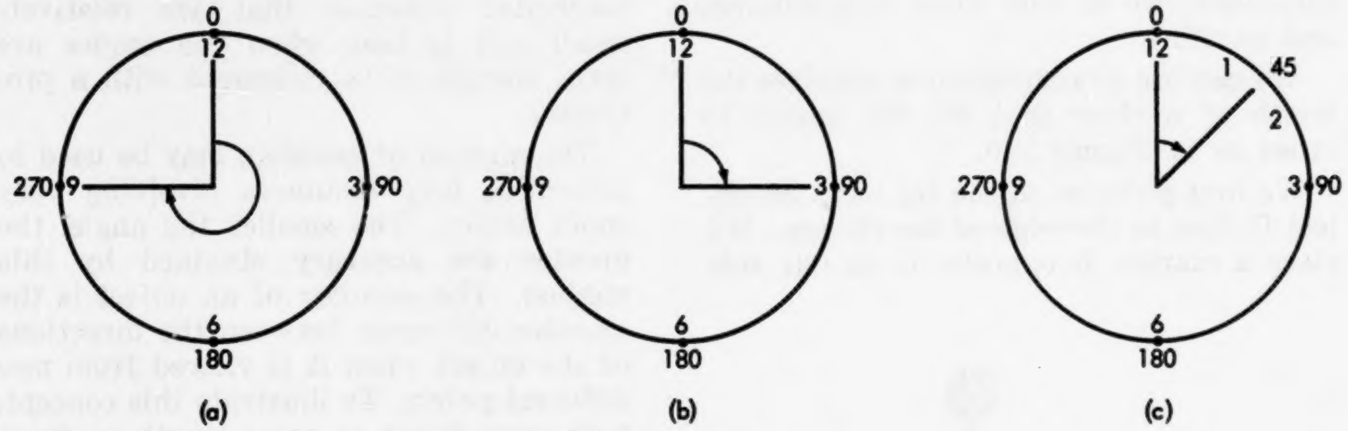
60
270
90
180
0
270
90
150
180

Figure 3–7


0
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270
9
3
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6
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0
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3
90
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180
0
12
1
45
2
270
9
3
90
6
180

(a) (b) (c)

Figure 3–8

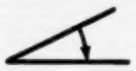
should not be confused:

1 revolution = 360° (degrees)
1 degree = 60′ (minutes)
1 minute = 60″ (seconds)

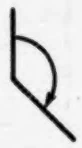
Astronomers need to measure angles accurately to the nearest hundredth of a second. The Orbiting Astronomical Observatory, OAO, will be able to maintain its position with an accuracy of 0.1 seconds. Accuracy and precision in measuring angles are extremely important in determining distances on the earth or in the sky.

3–2 Exercises—Direct Angular Measurement

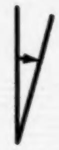
1. Measure each angle in the figures below.



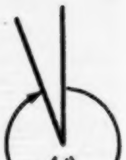
(a)



(b)



(c)



(d)

2. Construct angles with measure:
(a) 112.5° (b) 47° (c) 210′
(d) 360,000″

3. Sketch a clock with one hand on 12 and show:
   (a) 1½ hours (b) 7⅓ hours (c) 5 hours (Note that a real clock would not have one hand exactly on 12 in each case but for our purposes we will consider it there). Label each figure in hours and degrees both.
4. Make an array or table which will show each measure expressed as a number of degrees, as a number of minutes, and as a number of seconds:
   (a) 15° (b) 40′ (c) 5′ (d) 30″

3–3 Indirect Measurement

The preceding sections have shown some of the methods of measuring distances and angles directly with rule or protractor. However we are not always able to make measurements directly. We cannot place markers in space, stop the moon in its orbit, or stretch a tape measure around the earth.

Many methods have been developed to make measurements indirectly. We will be concerned with a number of these and introduce two at this time, triangulation and parallax.

We can use triangulation to measure the width of a river that we are unable to cross as is Figure 3–9.

We first pick out on the far bank an object C close to the edge of the stream. We place a marker B opposite C on our side of the river. From B we lay out a baseline $\overline{AB}$ which is perpendicular to $\overline{BC}$ and measure off a given distance; suppose we make $\overline{AB}$ 300 feet long. We use a transit to measure angle BAC, suppose $\angle BAC = 75°$. Since $\angle ABC = 90°$, $\triangle ABC$ is a right triangle. The distance $\overline{BC}$ across the stream can be computed as follows: (You will need to use the table of trigonometric functions on page 184).

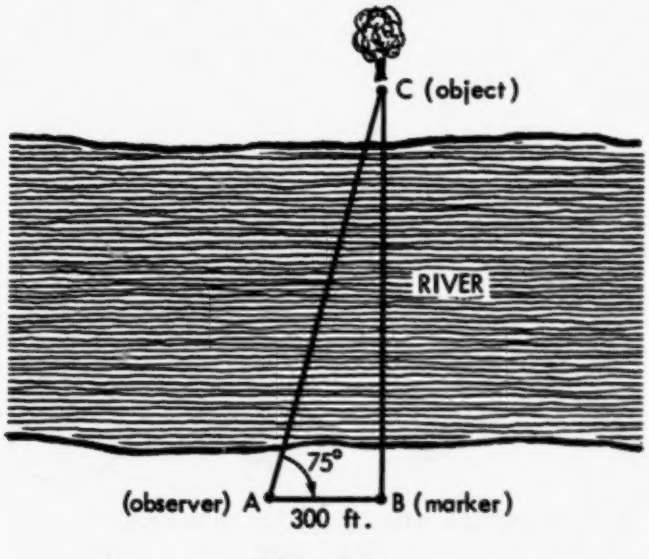


Figure 3–9

$$\frac{\overline{BC}}{\overline{AB}} = \tan \angle BAC$$

$$\begin{aligned}\overline{BC} &= \overline{AB} \text{ x } \tan \angle BAC\\ &= 300 \text{ x } \tan 75°\\ &= 300 \text{ x } 3.732 \text{ (Refer to the table of trigonometric functions)}\end{aligned}$$

$$\overline{BC} \approx 1{,}119.6 \text{ feet}$$

Then $\overline{BC}$ is about 1,120 feet. Remember that the value for the tan 75° was taken from the table of values for tangents of angles on page 184. Refer to this table whenever you need to.

This method of triangulation is used to determine distances that are relatively small and is best when the angles are large enough to be measured with a protractor.

The method of parallax may be used to determine long distances involving very small angles. The smaller the angle, the greater the accuracy obtained by this method. The *parallax* of an object is the *angular difference* between the directions of the object when it is viewed from two different points. To illustrate this concept, hold your finger at arms length in front of your nose and look toward a distant point or object such as the corner of the blackboard.

1. Close your right eye. What is your finger in front of now?

2. Close your left eye, and open your right eye. What is your finger in front of now?

3. Alternately open and close your eyes. Your finger should appear to "jump" back and forth as you alternately open and close your eyes.

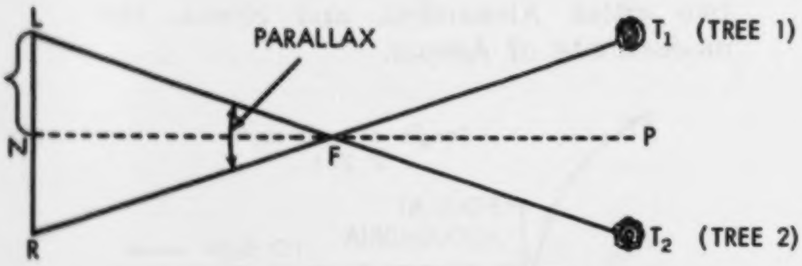


Figure 3–10

By using this very simple phenomenon, we are able to illustrate methods for measuring distances with accuracy. Consider Figure 3–10 where NFP is the line of sight using both eyes, LFT$_2$ is the line of sight using the left eye only, and RFT$_1$ is the line of sight using the right eye only.

If we knew the distance LN, and the distance NF, we could calculate the *parallax:* that is, ∠LFR. Notice that the vertical angles T$_1$FT$_2$ and LFR are congruent.

As in Figure 3–11 suppose that

$$\overline{LN} = 1.5 \text{ inches and}$$
$$\overline{NF} = 21 \text{ inches.}$$

$$\text{Then: } \tan \angle LFN = \frac{\overline{LN}}{\overline{NF}} = \frac{1.5}{21} \approx 0.07$$

$$\angle LFN \approx 4^\circ$$

This angle (∠LFN) is called the *horizontal parallax*; that is, one half of the parallax, ∠LFR in Figure 3–10. Notice in Figure 3–10 that

$$\angle LFN = \angle RFN = \angle T_1FP = \angle T_2FP$$

and that if we were to measure the horizontal parallax, then we would know the measures of all four of these angles and could determine the distance $\overline{NF}$.

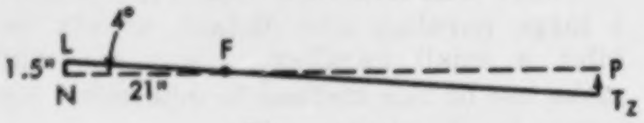


Figure 3–11

$$\tan \angle LFN = \frac{\overline{LN}}{\overline{NF}}$$

$$\overline{NF} = \frac{\overline{LN}}{\tan \angle LFN} = \frac{1.5}{\tan 4^\circ} \approx \frac{1.5}{0.07}$$

$$\approx 21 \text{ inches.}$$

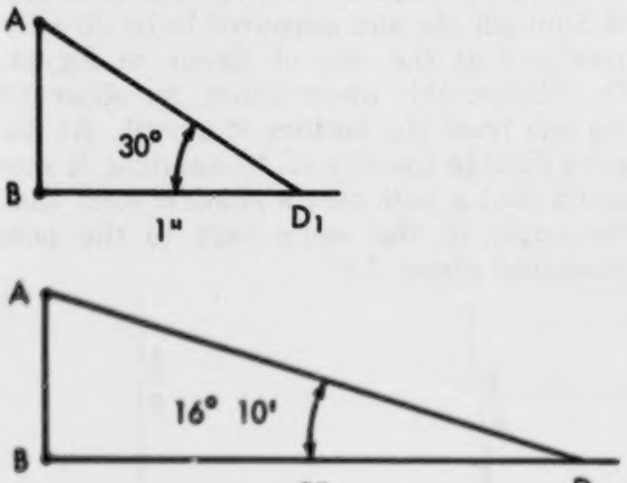
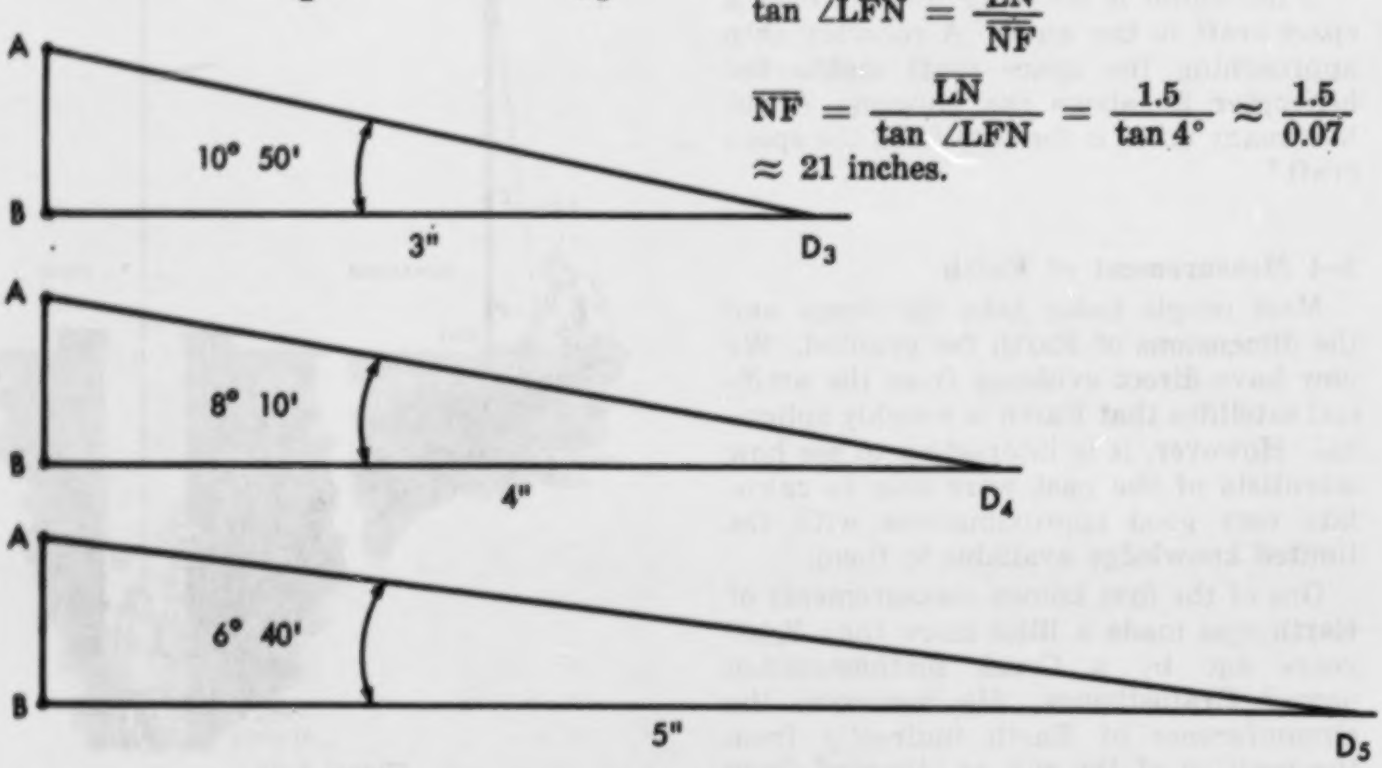



Figure 3–12

Here's another experiment, again place your finger in front of your nose. While alternately opening and closing your eyes, move your finger away from and then toward your nose. The closer your finger, the larger the "jump" and hence the larger the parallax. The further away your finger is, the smaller the "jump" hence the smaller the parallax. Figure 3–12 and Table 3–3 show the relation between parallax and distance for the given baseline $\overline{AB}$.

TABLE 3-3

Point	Parallax	Horizontal Parallax	Distance (inches)
D_1	60°	30°	1
D_2	32°20′	16°10′	2
D_3	21°40′	10°50′	3
D_4	16°20′	8°10′	4
D_5	13°20′	6°40′	5

Notice that relatively close objects show a large parallax and distant objects exhibit a small parallax. Later we shall make use of this method to determine distances to objects in space.

3–3 Exercise—Indirect Measurement

A helicopter is hovering 300 feet over a space craft in the water. A recovery ship approaching the space craft sights the helicopter 3° above the horizon. About how many miles is the ship from the space craft?

3–4 Measurement of Earth

Most people today take the shape and the dimensions of Earth for granted. We now have direct evidence from the artificial satellites that Earth is roughly spherical. However, it is interesting to see how scientists of the past were able to calculate very good approximations with the limited knowledge available to them.

One of the first known measurements of Earth was made a little more than 2,000 years ago by a Greek mathematician named Eratosthenes. He measured the circumference of Earth indirectly from the position of the sun as observed from two cities Alexandria and Syene, the modern city of Aswan.

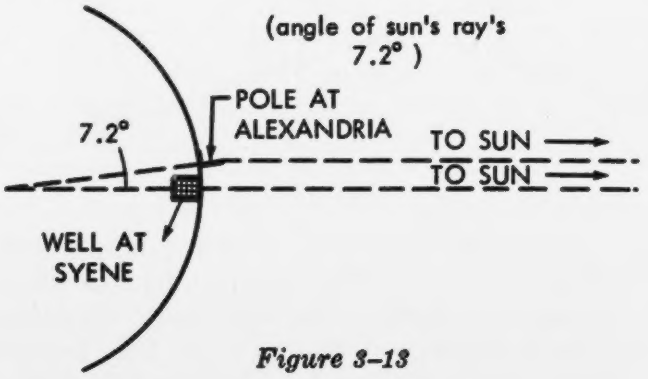


Figure 3–13

As the story is related today, Erastosthenes noted that at noon on the first day of Summer the sun appeared to be directly overhead at the city of Syene in Egypt. To confirm this observation, he observed the sun from the bottom of a well. At the same time in the city of Alexandria, it was noted that a pole cast a shadow such that the angle of the sun's rays to the pole measured about 7.2°.

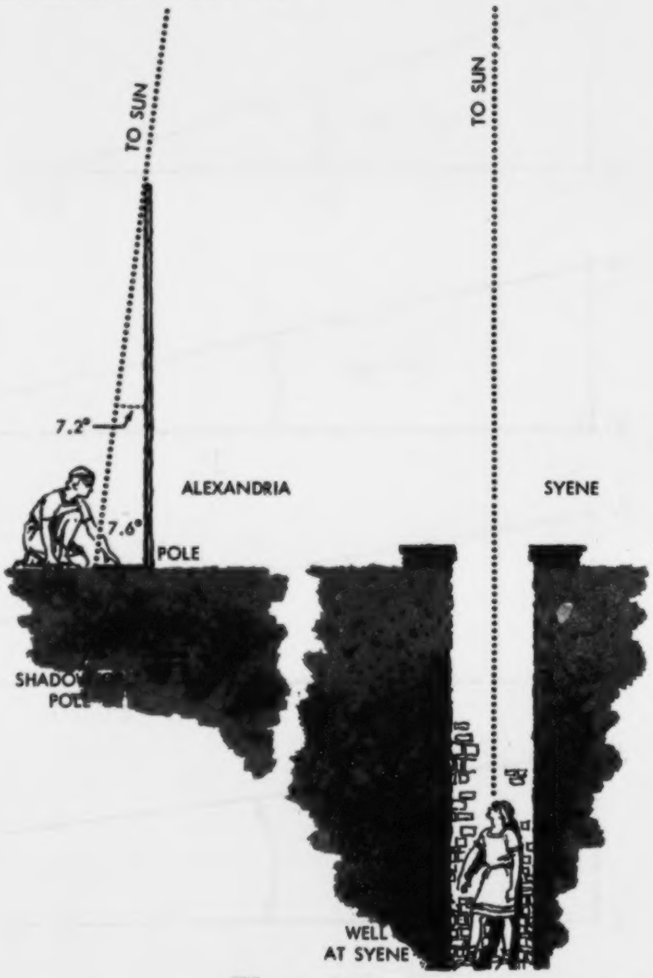


Figure 3–14

By reasoning that the sun's distance from Earth was a huge distance, Eratosthenes assumed that the rays of the sun striking the two cities were parallel (Figure 3–13). Using methods of his day, he carefully measured the distance between the well in Syene and the pole in Alexandria. The distance was obtained in the units of his day as 5,000 stadia. The actual distance a stadia represented is not known, but best estimates today place 10 stadia equal to about 1 mile. Eratosthenes then set up the following proportion to obtain the circumference of the earth:

$$\frac{5{,}000 \text{ stadia}}{7.2^\circ} = \frac{\text{circumference of Earth}}{360^\circ}$$

The figure obtained for the circumference of a great circle of Earth was about 250,000 stadia or 25,000 miles. This figure is very close to today's accepted value. The diameter of Earth can be found as:

$$d = \frac{\text{circumference}}{\pi} = 80{,}000 \text{ stadia or } 8{,}000 \text{ miles}$$

The radius is found by dividing the diameter by 2. The radius of Earth would be found as about 40,000 stadia or 4,000 miles. He obtained these figures 2,000 years before man finally had acceptable proof that Earth was essentially round and not flat. These measurements are based upon an assumption that Earth is a perfect sphere. More modern methods of measuring Earth have turned to the stars for greater accuracy. Terrestrial triangulation is one of the modern methods of determining distances on Earth.

We now consider Figure 3–15 and use a triangulation method as in Section 3–3 except that very precise measurements are made. The baseline $\overline{AB}$ can be measured to the nearest millionth of an inch; angles BAC and ABC can each be measured to the nearest 2 seconds of a degree. Then the distance $\overline{AC}$ can be computed by methods that are usually studied in high school trigonometry. Then $\overline{AC}$ can be used to measure $\overline{CD}$ which is indicated by the dotted line. Continuing by using the dotted line in this manner $\overline{AF}$, $\overline{AJ}$, and $\overline{JF}$ can each be measured to a high degree of accuracy.

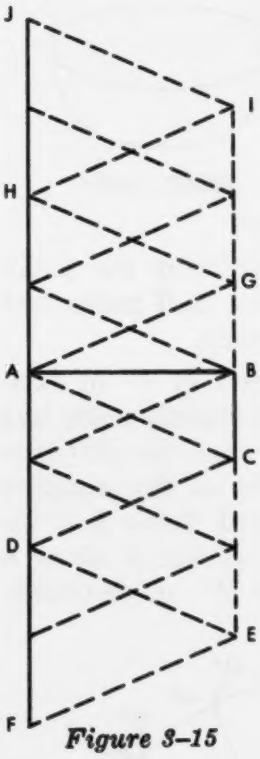


Figure 3–15

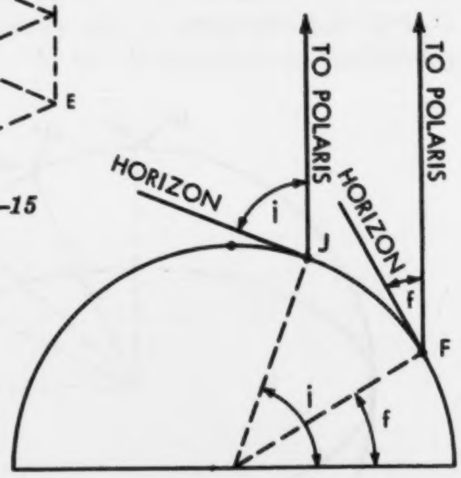


Figure 3–16

Observation of the North Star provides a very accurate method for determining Earth's measurements.

The North Star, Polaris, is considered to be directly above the North Pole. The altitude of Polaris above the horizon indicates the Observer's position on Earth. From points J and F, Figures 3–15 and 3–16, the altitude of Polaris, ∠j and ∠f, is determined. ∠j and ∠f are equal to the north latitude of J and F. By subtracting the differences in latitudes of the points, the length of JF can be determined in degrees. From this and the distance from J to F in miles, the circumference of Earth can be determined with greater accuracy than by the method of Eratosthenes.

This method of measuring Earth first indicated that it is not a perfect sphere but flattened at the poles, like an oblate

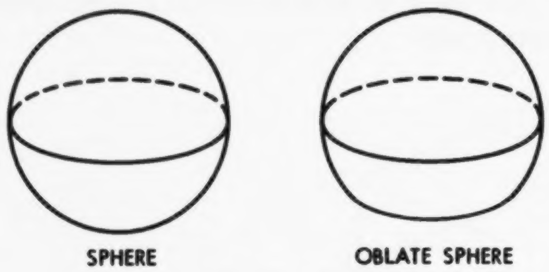


Figure 3–17

sphere (Figure 3–17). Actually the polar diameter of Earth is about 26.7 miles less than the equatorial diameter.

The distance represented in 1° of latitude at the north or at the south poles has been found to be greater than the distance represented by 1° latitude at the equator of Earth. Figure 3–18 and Table 3–4 indicate the changes in the distance along a meridian represented by 1° of latitude.

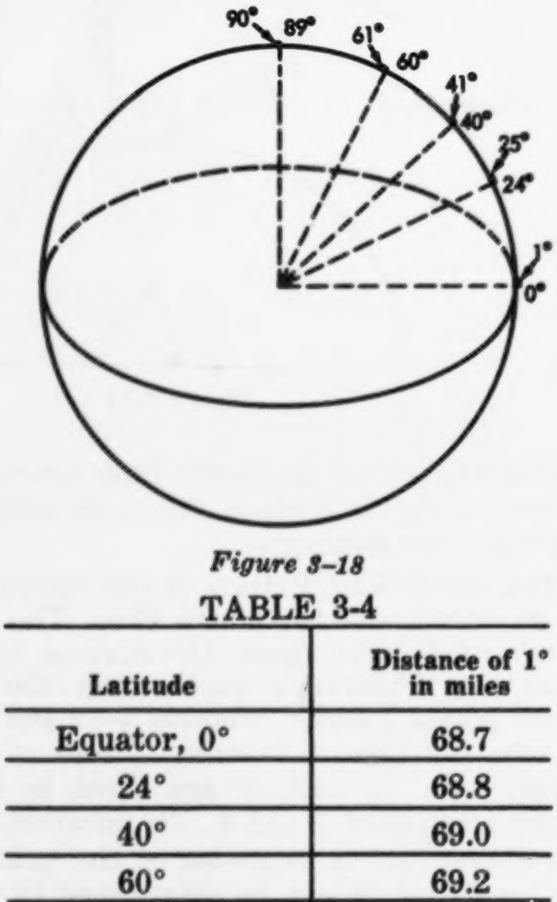


Figure 3–18

TABLE 3-4

Latitude	Distance of 1° in miles
Equator, 0°	68.7
24°	68.8
40°	69.0
60°	69.2
N or S Pole (90°)	69.4

Artificial satellites have enabled man to make rather precise determinations of the size and shape of Earth. Previously, the radius of Earth could be determined only within 700 yards, that is between 3,963.188 and 3,963.250 miles. Today particularly with the use of Vanguard I, the analysis of the orbit of the satellite places the average radius of Earth at the most probable value of 3,963.210 miles. Prior to the artificial satellites, the limitations of terrestrial triangulation and cellestial observation limited the accuracy of the distance between the continents of America and Europe to within a mile. With the worldwide tracking stations and electronic computers, the orbits of satellites can be determined with great accuracy. By using the position of the satellite in its orbit, scientists are now able to compute the positions of islands in the middle of the Pacific ocean to within 25 yards.

Vanguard I has also enabled man to refine his ideas pertaining to the shape of Earth. In 1958, scientists studying the orbit of Vanguard I determined that Earth was not symmetricaly oblate, but the southern hemisphere actually bulged more than the northern hemisphere. This indicated that Earth was not spherical, not oblate, but was somewhat pear-shaped!

These differences seem really small, and are not apparent to *us*, but these and other minute differences are of extreme importance to interplanatory probes, such as the Mariner IV flight to Mars and to our ideas of the structure and evolution of Earth.

3–4 Exercises—Measurement of Earth

The angular separation between the East and West coast of the United States is about 44°. About what is the distance in miles of this angular separation? (Use 3,970 miles for the radius of Earth.)

3–5 Altitude of a Model Rocket

The determination of the altitude of a model rocket is a relatively simple exercise in triangulation. We will not attempt to plot the flight path of the rocket but simply to determine its altitude *a* at its highest point above the surface of the Earth.

Consider the data in Figure 3–19. The problem could be solved by this method only if the rocket rose vertically from the launch site and "peaked" directly above the launch site.

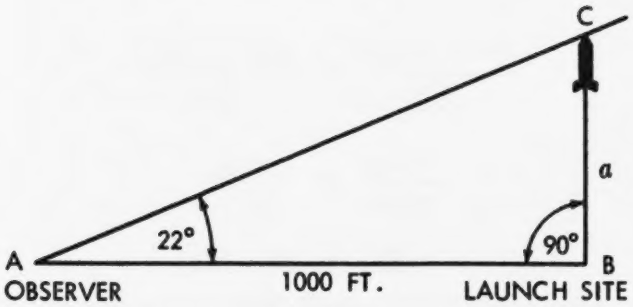

C
a
22°
90°
A
B
OBSERVER
1000 FT.
LAUNCH SITE

Figure 3–19

Here is a solution to the problem as illustrated in Figure 3–19. Prior to launch, the distance from A to B is measured and found to be 1000 feet. After launch, the flight of the rocket is observed from position A with a transit and the highest elevation of the rocket is noted as 22°. The altitude $\overline{BC}$ of the rocket may be determined by using the formula for the tangent of ∠BAC. (The tangent formula is explained in Section 1–7.)

$$\frac{\overline{BC}}{\overline{BA}} = \tan \angle BAC$$

$$\overline{BC} = \overline{BA} \times \tan 22^\circ$$

$$\overline{BC} \approx 1000 \times 0.404$$

$$a = \overline{BC} \approx 404 \text{ feet}$$

The preceding problem is a theoretical example. In practice the rocket does not rise directly over the launch site, but follows a curved path to its greatest altitude. Primary factors which effect the flight of a model rocket are:

(1) Aerodynamics of the rocket.
(2) Wind velocity at various altitudes.
(3) Change in the center of gravity of the rocket due to the burning of fuel.
(4) Angle of launch.

A practical approach in solving this problem has been worked out by the National Association of Rocketry. The following method is used by the Association to determine the altitudes of model rockets in contests. The method requires two observers each equipped with a transit which will measure direction horizontally and elevation vertically.

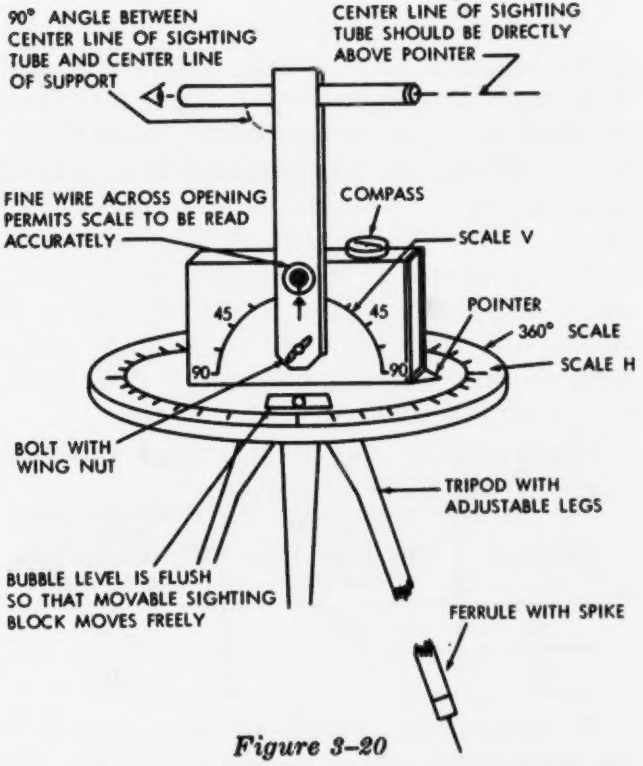

90° ANGLE BETWEEN CENTER LINE OF SIGHTING TUBE AND CENTER LINE OF SUPPORT
CENTER LINE OF SIGHTING TUBE SHOULD BE DIRECTLY ABOVE POINTER
FINE WIRE ACROSS OPENING PERMITS SCALE TO BE READ ACCURATELY
COMPASS
SCALE V
45
45
90
90
POINTER
360° SCALE
SCALE H
BOLT WITH WING NUT
TRIPOD WITH ADJUSTABLE LEGS
BUBBLE LEVEL IS FLUSH SO THAT MOVABLE SIGHTING BLOCK MOVES FREELY
FERRULE WITH SPIKE

Figure 3–20

Scale H in Figure 3–20 gives the angle in degrees horizontally from the baseline to the position of the rocket. Scale V gives the angle vertically in degrees from horizontal.

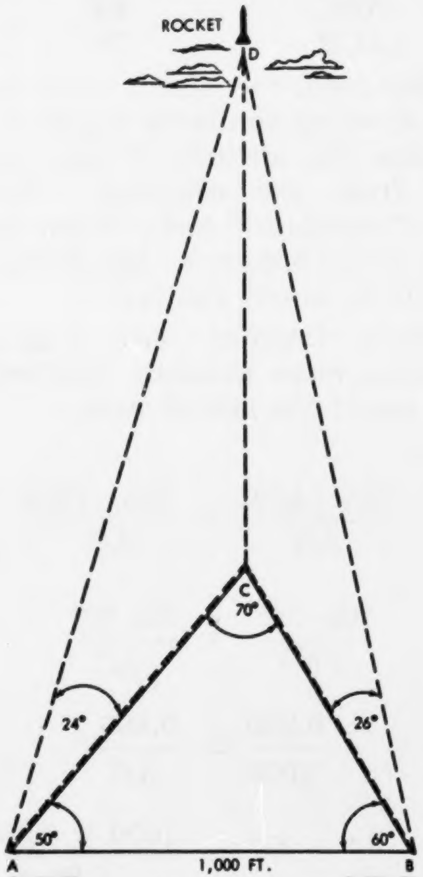

ROCKET
D
C
70°
24°
26°
50°
60°
A
OBSERVER
1,000 FT.
B
OBSERVER

Figure 3–21

Observers A and B are positioned prior to the launch. The distance between A and B is carefully measured to be 1,000 feet. At launch the horizontal scale reads 0° direction along the baseline and the vertical scale reads 0°; both observers observe the flight with a transit much as you would watch an airplane taking off. At its highest point, D, the rocket gives off a puff of smoke and the observers tighten the wing nuts on the transits (Figure 3–20) to fix the angle on scale V.

Observers A and B can read the two angles from their transits.

Observer	Scale H Direction from baseline	Scale V Elevation from horizontal
A	50°	24°
B	60°	26°

To obtain the altitude of the rocket $\overline{DC}$, we first must solve the triangle ABC for the distances $\overline{AC}$ and $\overline{BC}$. From Figure 3–21, we obtain the following data:

$\overline{AB}$	1000 feet
$\angle CAB$	50°
$\angle ABC$	60°
$\angle ACB$	70°

At this point the reader could construct a scale drawing similar to Figure 3–21 and determine the altitude of the rocket directly from the drawing. (Suggested scale: 1" equals 200 feet.) From the drawing, we would determine the altitude of the rocket to be about 410 feet.

Students familiar with trigonometry may make more accurate determinations by the use of the law of sines:

$$\frac{\text{Sin } \angle ACB}{\overline{AB}} = \frac{\text{Sin } \angle CBA}{\overline{AC}}$$

$$\frac{\text{Sin } 70^\circ}{1000} = \frac{\text{Sin } 60^\circ}{\overline{AC}}$$

$$\frac{0.940}{1000} \approx \frac{0.866}{\overline{AC}}$$

$$\overline{AC} \approx \frac{1000 \times 0.866}{0.94}$$

$$\overline{AC} \approx 921 \text{ feet}$$

$$\frac{\text{Sin } \angle ACB}{\overline{AB}} = \frac{\text{Sin } \angle CAB}{\overline{BC}}$$

$$\frac{0.940}{1000} \approx \frac{0.766}{\overline{BC}}$$

$$\overline{BC} \approx 815 \text{ feet}$$

This determines each observer's distance from point C, directly below the highest point in the rocket's flight. Each observer can now use the tangent formula to calculate the altitude of the rocket.

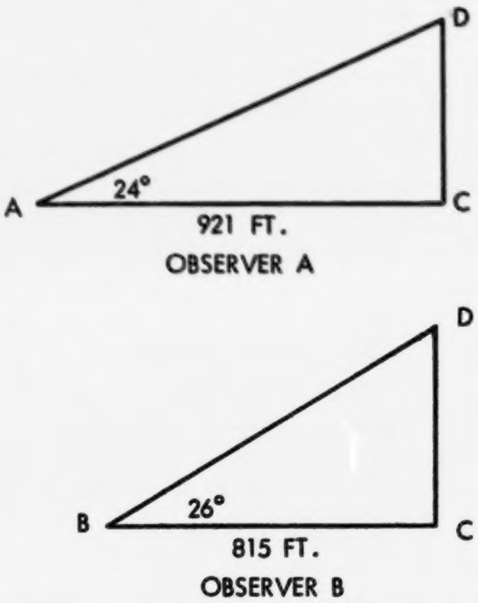


Figure 3–22

Observer A $\quad \tan 24^\circ = \dfrac{\overline{DC}}{\overline{AC}}$

$$0.445 \approx \frac{\overline{DC}}{921}$$

$$\overline{DC} \approx 410 \text{ feet}$$

Observer B $\quad \tan 26^\circ = \dfrac{\overline{DC}}{\overline{BC}}$

$$0.488 \approx \frac{\overline{DC}}{815}$$

$$\overline{DC} \approx 398 \text{ feet}$$

As calculated by observer A, the altitude of the rocket is 410 feet and by B the altitude of the rocket is 398 feet. The average altitude of the rocket is 404 feet. As outlined in the rules of the National Association of Rocketry 404 feet is the accepted value. To have a successful and competitive flight under the rules of the NAR, the altitudes calculated by Observers A and B must agree within 10% of the average value. In our example 10% of 404 feet is about 40 feet and both calculated values are within 10% of the average value.

Exercises

See if you can complete the following table, indicating the altitudes obtained by each observer, the average altitude, and if the results were acceptable.

Observer A				**Observer B**				
d (ft)	**Scale H**	**Scale V**	**Altitude a**	**Scale H**	**Scale V**	**Altitude DC**	**Average Altitude**	**Acceptable (Yes) (No)**
1000	30°	30°		60°	45°			
1000	40°	40°		65°	50°			
1000	37°	48°		68°	60°			
1000	52°	31°		70°	32°			
1000	12°	56°		104°	81°			
1000	64°	73°		43°	68°			

Note: This problem is applicable to computer programming. See Section 6–5 Exercise 2.

NOTE TO STUDENTS

Several sections of Chapter III involve detailed computations. However, if you simply read the material, you will discover some of the interesting methods astronomers use to study the stars. It is not necessary that you perform all the computations. You should try them and complete as many as you can.

Remember that you need to "see" a three dimensional model in your mind as you look at the drawings. You may wish to construct some models using the diagrams as guides. When you have done this, go back and try some of the computations that are suggested.

3–6 Earth as Viewed from a Satellite

The altitude of a satellite may be determined in much the same manner as that of a rocket. However it is necessary to use a very fine system of tracking stations and electronic computers to determine the precise orbit and altitude as used in the field of communication and weather satellites. The altitude of the satellite also determines the area of Earth that may be photographed or "spanned" by a radio signal.

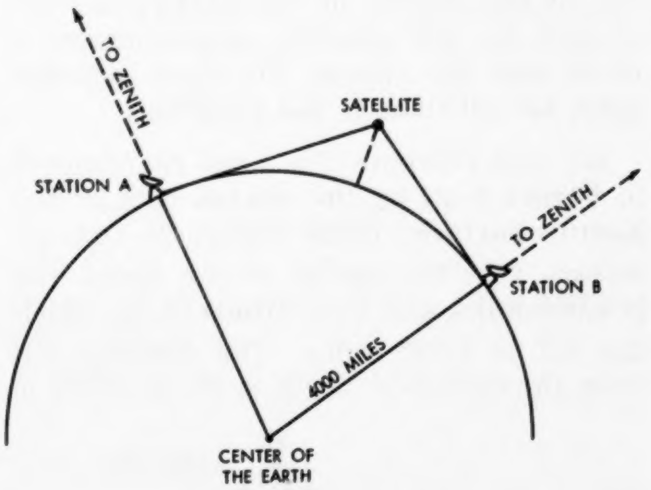


Figure 3–23

Figure 3–23 is drawn to scale. On our scale, the Earth's radius is equal to 4,000 miles. We are able to determine the altitude of the satellite as ¼ of Earth's radius or 1,000 miles. If we know the altitude of the satellite, we are able to determine the area of Earth that it could "span." As the satellite's altitude increases, the area of Earth viewed from the satellite increases (Figure 3–24).

Figure 3–24 illustrates the appearance of the portion of Earth viewed by satellites at different altitudes. Notice that the out-

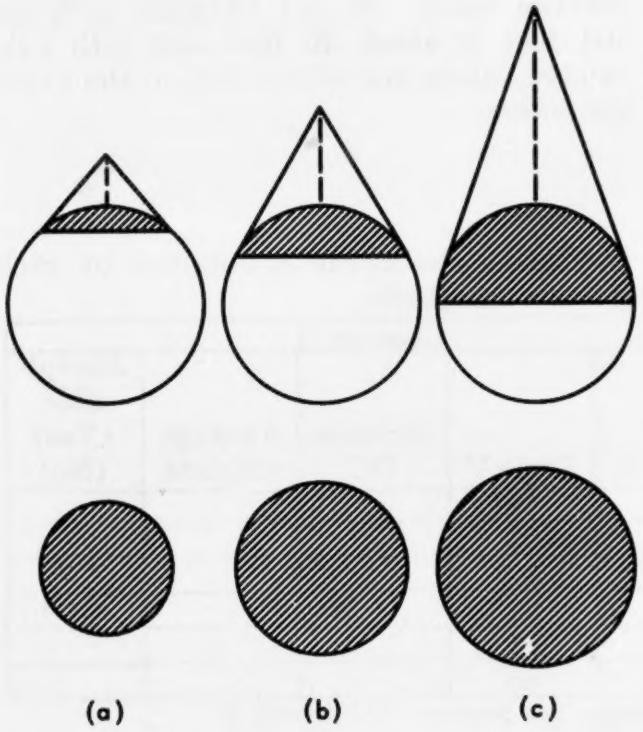


Figure 3–24

line of the portion of the Earth's surface viewed by the satellite approximates a circle and the size of the circle depends upon the altitude of the satellite.

We can compute the area represented in Figure 3–24 by the shaded part of the Earth's surface. From Figure 25 (a), we assume that the radius of the Earth $\overline{BE}$ is 4,000 miles and the altitude of the satellite $\overline{AE}$ is 1,000 miles. The distance $\overline{AB}$ from the center of Earth to the satellite is 5,000 miles. The limit of the satellite's view is represented by a line such as $\overline{AC}$ that is tangent to the surface of the Earth at C and forms a right angle with the radius of Earth $\overline{BC}$. We wish to determine the height $\overline{DE}$ of the shaded zone of Earth. Triangle ABC is a right triangle. We know two sides of triangle ABC; side $\overline{AB}$, 5,000 miles, and side $\overline{BC}$, 4,000 miles.

$$\cos e = \frac{\overline{BC}}{\overline{AB}} = \frac{4{,}000}{5{,}000} = 0.800$$

We are now able to determine $\overline{DB}$. Triangle DBC is a right triangle;

$$\overline{DB} = (\cos e) \times \overline{BC} = 0.800 \times 4000$$
$$= 3{,}200 \text{ miles}$$

We know that the radius $\overline{BE}$ of Earth is 4,000 miles. Then, since $\overline{BD} + \overline{DE} = \overline{BE}$ and $\overline{BD} = 3{,}200$ miles, we have $\overline{DE} = 4{,}000 - 3{,}200 = 800$ miles. The height $\overline{DE}$ of the zone of Earth is about 800 miles. The formula for the area of the zone of a sphere is $A = 2\pi rh$, where r is the radius of the sphere and h is the height of the zone (as in Section 1–5). The area of the zone that we have been considering is about 20,096,000 square miles since

$$A = \pi 2\, rh$$
$$\approx 2 \times 3.14 \times 4000 \times 800 = 20{,}096{,}000$$

If we wanted to know the part of the Earth's surface viewed by the satellite, we could determine this using the ratio of the

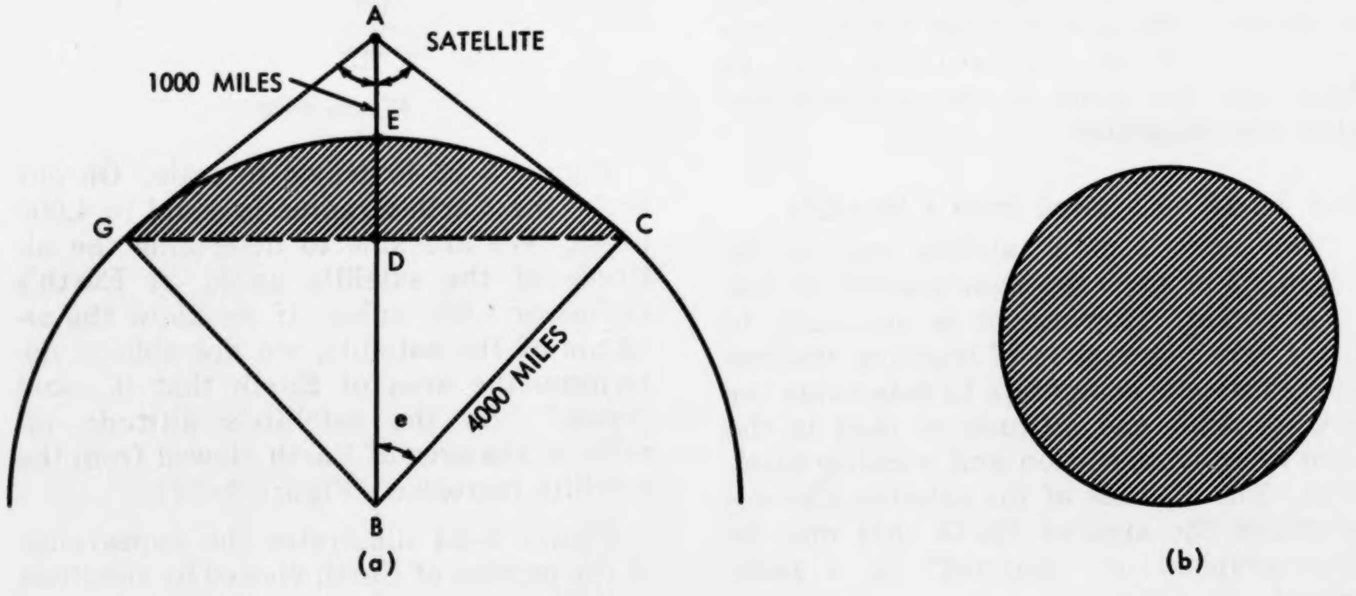


Figure 3–25

area of the zone to the area of the surface of Earth:

$$\frac{20,096,000}{200,960,000} = \frac{1}{10} = 10\%$$

The satellite views about 10% of the Earth's surface.

3-6 Exercises—Earth as Viewed from a Satellite

1. Complete as many of the blanks in the following table as you can.

	Altitude of the satellite (miles)	Visible surface (square miles)	Part of Earth's surface visible
(a)	200	---------	----
(b)	1000	20,096,000	10%
(c)	4000	---------	----
(d)	5000	---------	----

Note: This problem is applicable to computer programming. See Chapter 6–3, Exercises 3 and 4.

2. How many satellites at an altitude of 5,000 miles would be necessary to completely photograph the equator of Earth at a given time?
3. In Figure 3–25a what is the approximate distance along the surface of the Earth between G and C in miles?

3–7 Distance to the Moon

Although the distance to the moon is not the basic unit of measurement for astronomers, it is extremely important to us when related to determining the size and shape of Earth. Since the distance from Earth to the moon varies, we will use mean values to give us an approximation of the moon's distance.

We are able to make this calculation by the parallax method using the diameter of Earth as a baseline. This is referred to as the geocentric parallax, Earth-centered method.

From Figure 3–26, we are able to identify the points as follows: The center of Earth is at O and $\overline{AB}$ is a diameter. A surface feature of the moon, such as a mountain or crater, is indicated by point L. Stars marked S_1 and S_2 are at a distance so great that they exhibit no observable parallax with a baseline as small as $\overline{AB}$. Lines of sight $\overline{AL}$ and $\overline{BL}$ are tangent to the surface of the moon at a crater or mountain.

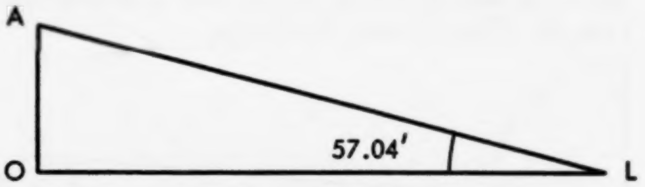


Figure 3–27

The moon's horizontal parallax has been measured many times and is found to have a mean value of 57.04 minutes; $\angle ALO \approx 57.04'$. We will use 3,963 miles as the Earth's radius; $\overline{AO} \approx 3,963$ miles. We treat $\overline{AO}$ as approximately equal to an arc of a circle with radius $\overline{OL}$ and central angle ALO. Then using radian measure as in Section 1–3,

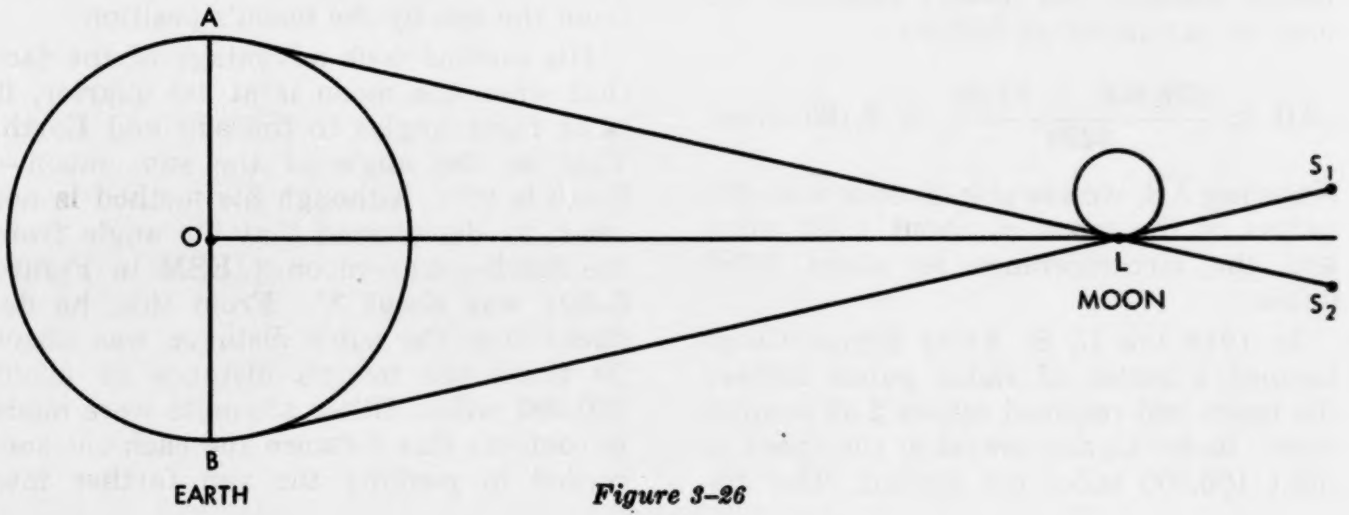


Figure 3–26

$$d = r\theta$$

$$\overline{AO} = \overline{OL} \times \angle ALO$$

when ∠ALO is measured in radians. Remember that

$$2\pi \text{ radians} = 360°,$$

$$1 \text{ radian} \approx \frac{360}{6.24} \approx 57.3°,$$

$$1 \text{ radian} \approx 57.3 \times 60 = 3438'.$$

The measure of ∠ALO in radians may be found by dividing its measure in minutes by 3438. Then:

$$\overline{AO} = \overline{OL} \times \frac{57.04}{3438}$$

$$\overline{OL} = \frac{3963 \times 3438}{57.04} \approx 238{,}900 \text{ miles.}$$

This is the moon's mean distance from the center of Earth. The moon's mean distance is the starting point for the calculation of other lunar statistics.

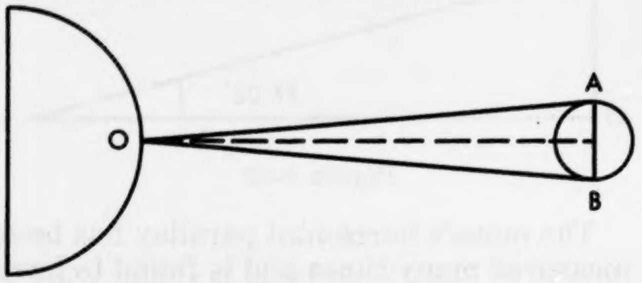


Figure 3–28

By reversing the parallax method and using the moon's mean distance from Earth we are able to calculate the diameter $\overline{AB}$ of the moon (Figure 3–28). Angle AOB has been measured many times and a value of 31.09′ determined. Using the radian method, the moon's diameter $\overline{AB}$ may be calculated as follows :

$$\overline{AB} \approx \frac{238{,}900 \times 31.09}{3438} \approx 2{,}160 \text{ miles}$$

Knowing $\overline{AB}$, we are able to determine the radius of the moon as about 1,080 miles and the circumference as about 6,780 miles.

In 1946 the U. S. Army Signal Corps beamed a series of radar pulses toward the moon and received echoes 2.56 seconds later. Radio signals travel at the speed of light 186,000 miles per second. The distance to the moon in its orbit at that time was determined to be about 238,500 miles.

3–7 Exercises—Distance to the Moon

1. If the mean distance to the moon is 238,900 miles, what is the approximate circumference of the moon's orbit of Earth?
2. If the moon orbits Earth once in 27½ days, what is its average speed in miles per hour?
3. The moon's orbit is not perfectly circular. Therefore, the moon's distance from Earth varies. The maximum horizontal parallax is given as 61.3 minutes and the minimum as 53.8 minutes. What is the moon's maximum and minimum distances from Earth?

3–8 The Yardstick of Space

The distance of Earth from the sun is one of the most important distances in all of astronomy. It is the basic unit of distance for the calculation of many distances in the universe. It is necessary that this distance be known as precisely as possible.

Historically man has attempted to determine the distance to the sun in many ways and has arrived at many different answers. The early Greek philosophers wrote of the sun as a fiery ball a few miles in diameter and a few thousand miles distant. The first attempt to determine accurately Earth's distance from the sun is credited to the Greek astronomer Aristarchus.

By noting the passage of the moon's phases, from new moon to 1st quarter to full moon and to 3rd quarter, Aristarchus attempted to determine Earth's distance from the sun by the moon's position.

His method took advantage of the fact that when the moon is at 1st quarter, it is at right angles to the sun and Earth. That is, the angle of the sun—moon—Earth is 90°. Although his method is not clear, he determined that the angle from the Earth—sun—moon (∠ESM in Figure 3–30) was about 3°. From this, he deduced that the sun's distance was about 20 times the moon's distance or about 700,000 miles. Other attempts were made to compute this distance and each one succeeded in pushing the sun farther into

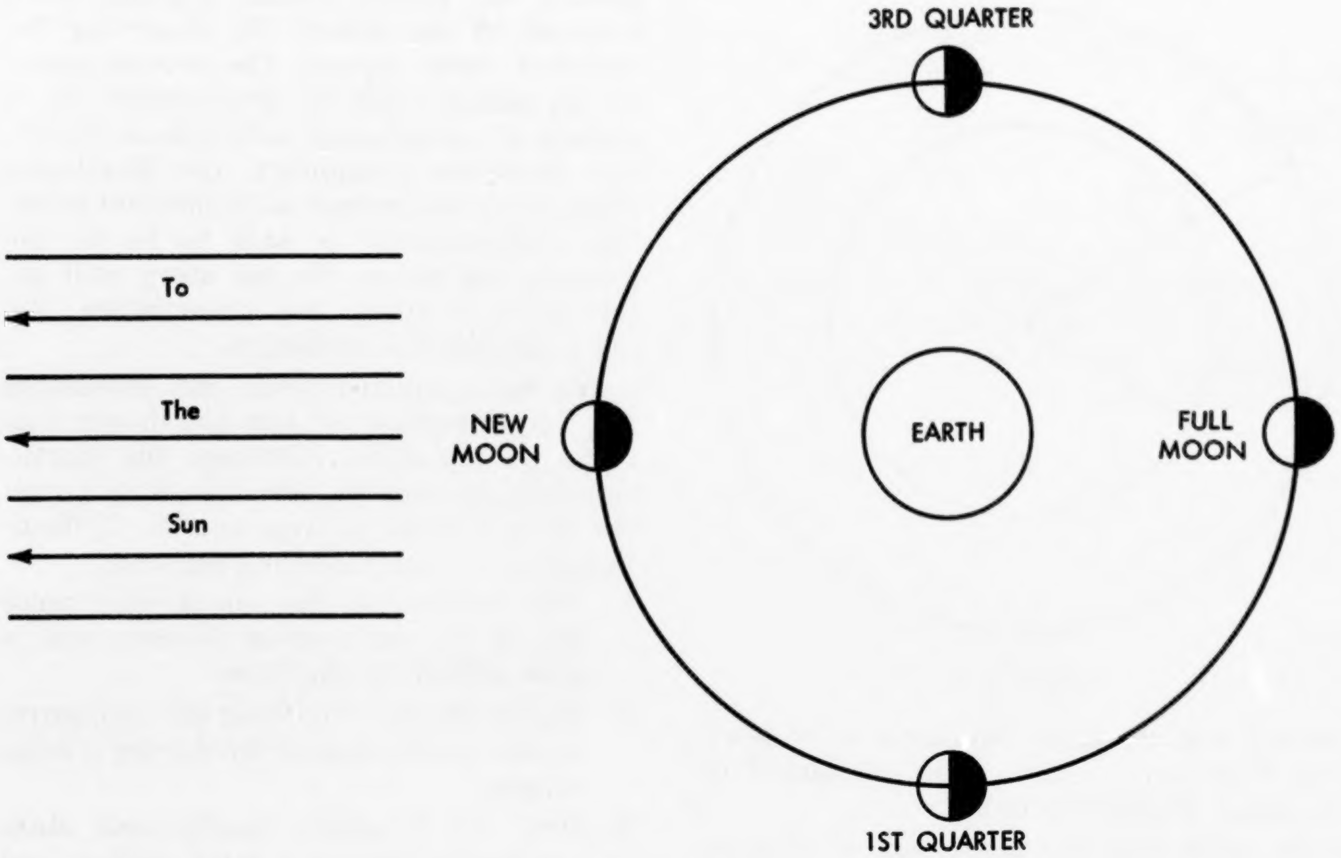


Figure 3–29

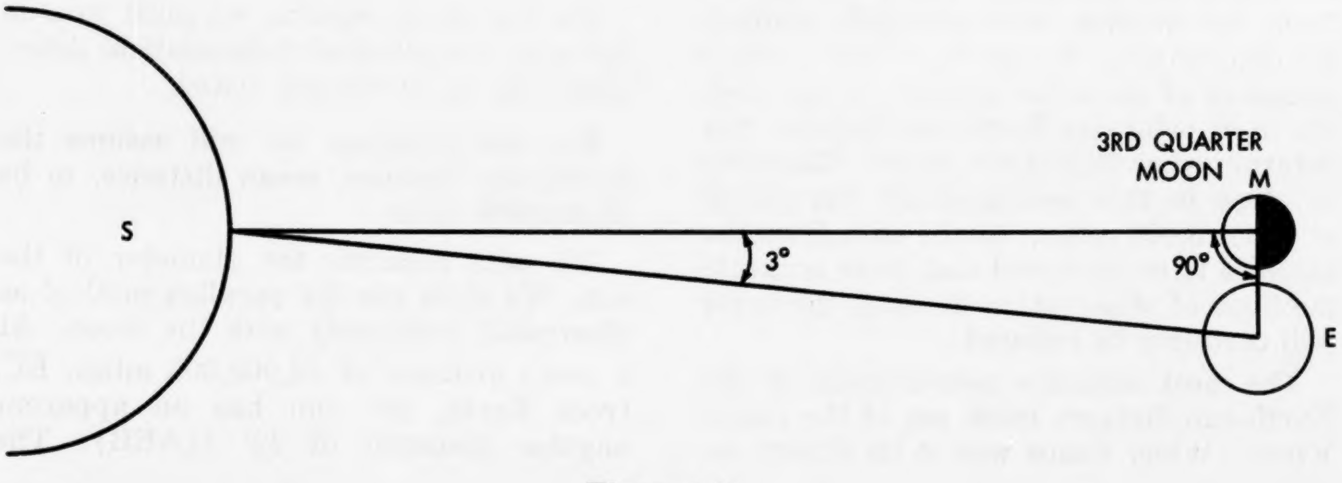


Figure 3–30

space. In 1769 astronomers, observing the passage of the planet Venus across the disc of the sun, calculated the earth-sun distance to be about 93,000,000 miles. This figure is close to the one we hold today. Due to the importance of this distance, astronomers continue to refine their measurements. Astronomers recognized that the Earth-sun distance varied depending upon Earth's position in its orbit. Remember that Earth's orbit is not circular but is elliptical. We will assume the Earth-sun distance to be the Earth's mean distance from the sun.

Making use of modern instruments, the first accurate measurement of the Earth-sun distance involved the asteriod Eros. Eros is one of thousands of "minor planets" that orbit the sun between Mars and Jupiter. The distance d from the Earth to Eros (Figure 3–31) was determined by parallax. The period of Eros (that is, the time of one orbit of the sun) was calculated. By using a relationship of

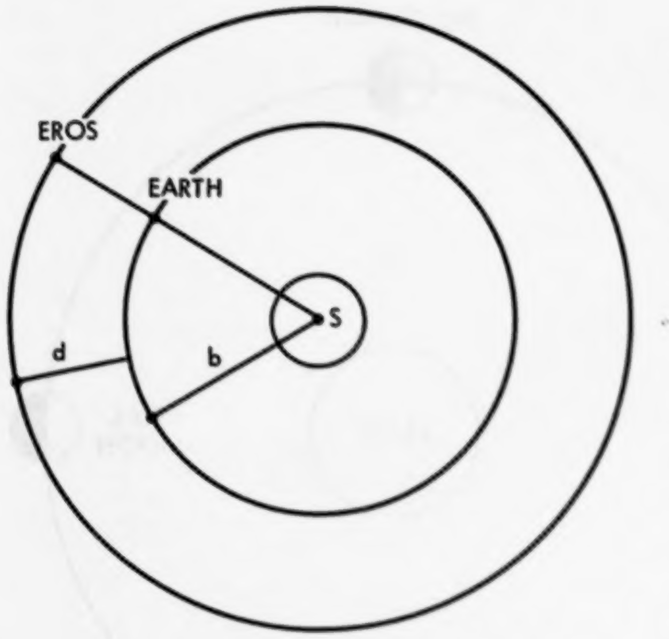


Figure 3–31

period and distance (Kepler's third law) the Earth-sun distance was calculated to be about 92,900,000 miles.

In 1960 with the launching of Pioneer V, man had a new method of determining the Earth-sun distance. The radio signals from the satellite were carefully studied. By determining the effects of the various members of the solar system on the satellite in its orbit, the Earth-sun distance was determined at 92,924,900 miles. The error involved in this measurement was placed at about 8,000 miles. As the data from the satellite is recalculated and more accurate methods of observation develop, the error will certainly be reduced.

The most accurate measurement of the Earth-sun distance made use of the planet Venus. When Venus was at its closest approach to Earth, radar signals were bounced off the planet. By observing the reflected radio signals, the precise speed of the planet could be determined. By a system of complicated calculations involving electronic computers, the Earth-sun distance is determined as 92,956,300 miles. The measurement is said to be to the nearest 300 miles. So the story will go, the more accurate the observation, the more precise the prediction.

We have omitted from our discussion the determination of the Earth-sun distance by parallax. Although the Earth-sun distance may be determined by parallax, it is subject to large errors. Difficulties arise for the following reasons:

1. The parallax of the sun is very small due to the sun's great distance and is also difficult to measure.
2. Due to the sun's brilliant disc, measurements can be made only during a solar eclipse.
3. Only the brightest background stars are visible during a solar eclipse, and very few could be viewed near the sun.

For the above reasons, we must turn to the more complicated mathematical determinations as previously listed.

For our purposes we will assume the Earth-sun distance, mean distance, to be 92,900,000 miles.

We next measure the diameter of the sun. We shall use the parallax method as illustrated previously with the moon. At a mean distance of 92,900,000 miles, $\overline{EC}$, from Earth, the sun has an apparent angular diameter of 32′ (∠AEB). The

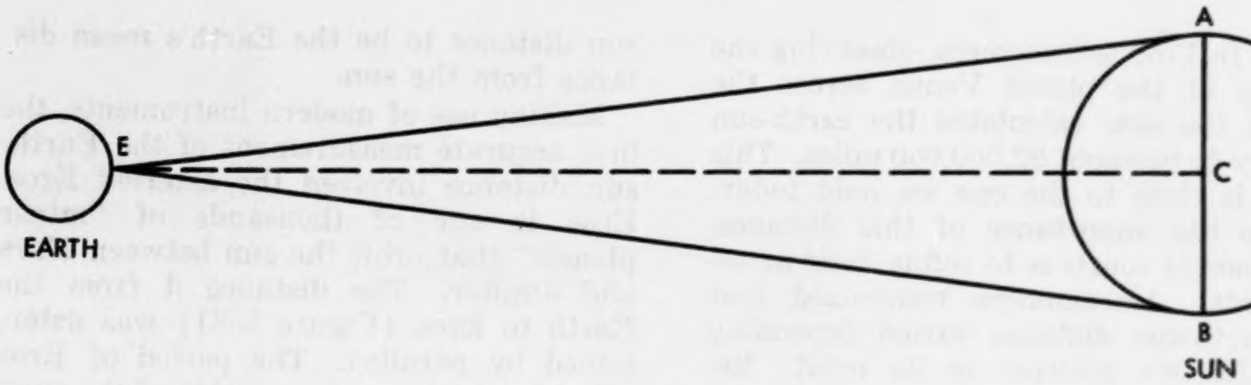


Figure 3–32

sun's diameter may be calculated by the radian method as follows:

$$\overline{AB} \approx \frac{92{,}900{,}000 \times 32}{3438} \approx 864{,}600 \text{ miles}$$

We could calculate the radius and the circumference of the sun by making use of this measure of the diameter.

Knowing the Earth's mean distance to the sun, we could calculate an approximate circular orbit for Earth. From this we could determine the mean velocity of Earth as it orbits the sun in a year.

The mean distance of the sun from Earth, 92,900,000 miles, is called by astronomers *1 astronomical unit.* The *astronomical unit,* A.U., is the unit used to measure distances to the stars. Thus we no longer need to concern ourselves with the large number, 92,900,000, to represent the Earth-sun distance, but may say simply 1 astronomical unit (1A.U.).

3–8 Exercises—The Yardstick of Space

1. Compute the distance travelled by Earth in one orbit. What is its average velocity in miles per hour?
2. The Earth's orbit is not circular. As a mean distance from the sun, 92,900,000 miles, the sun has an angular diameter of 32′ (Figure 3–32). At Earth's maximum distance of 94,500,00 miles, what is the sun's angular diameter?
3. The Earth's minimum distance to the sun is 91,500,000 miles. What is the sun's angular diameter at this distance?
4. If the diameter of the sun is 864,000 miles, how many times larger is the sun's diameter than the Earth's?
5. How many times larger in volume than the Earth is the sun?

3–9 The Inner Planets

Mary's	Violet	Eyes	Make	John	Stay	Up	Nights	Period
E	E	A	A	U	A	R	E	L
R	N	R	R	P	T	A	P	U
C	U	T	S	I	U	N	T	T
U	S	H		T	R	U	U	O
R				E	N	S	N	
Y				R			E	

To Help You Remember the Names of the Planets

The discovery of the motion and physical characteristics of the planets is a book by itself. We will concern ourselves with the problems associated with the construction of a scale drawing of the solar system. In this manner we might appreciate the dimensions of the system and what a tiny fraction of this system we occupy.

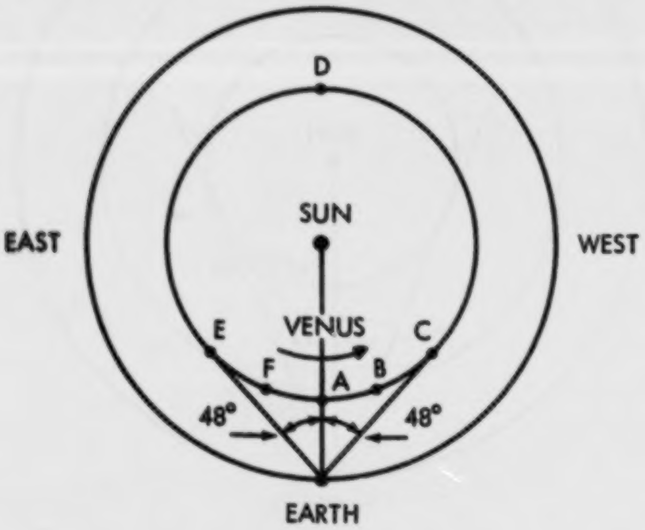


Figure 3–33

In the construction of our scale drawing of the solar system, we will separate the planets into two groups. The inner planets include Mercury, Venus, and Earth (Mary's Violet Eyes). The outer planets include Mars, Jupiter, Saturn, Uranus, Neptune, and Pluto (Make John Stay Up Nights Period). In our discussion of the planets, we will approximate orbits as circles, as in Section 2–4.

To place Venus in its orbit in our scale drawing, we refer to Figure 3–33. When Venus is directly between the sun and Earth, position A, it is not visible from

Earth due to the sun's bright light. However, Venus is closer to the sun and moves faster in its orbit than Earth moves in its orbit. As Venus moves from position A to B, it becomes visible from Earth. As it moves *toward* the west, (as viewed from Earth) it is visible each morning before sunrise. Each morning it is a little higher in the sky at sunrise. As seen from Earth, Venus finally reaches its greatest separation from the sun at position C, Figure 3–34.

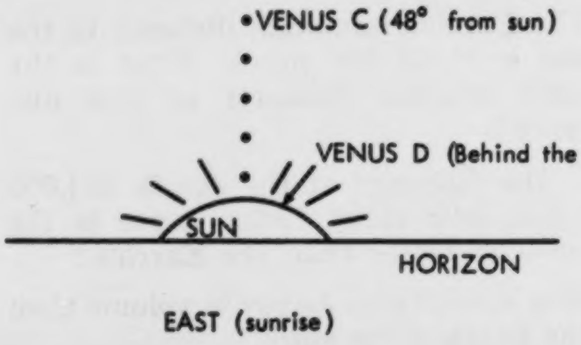


Figure 3–34

At its greatest separation from the sun, the angle between the sun and Venus is about 48°. After Venus passes position C in its orbit. each morning Venus appears from the Earth to be closer to the sun. Finally it becomes invisible to us as it passes behind the sun, position D. As Venus continues in its orbit it slowly becomes visible to us just after sunset, Figure 3–35.

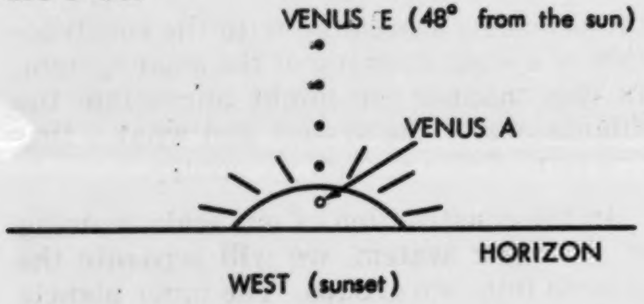


Figure 3–35

Each evening Venus appears to be just a little bit higher in the sky at sunset. Finally when Venus reaches position E in its orbit, it is again at its greatest separation from the sun. Each evening Venus appears from Earth to be closer to the sun as it continues in its orbit and returns to position A. We now are able to place the orbit of Venus in our scale drawing of the solar system.

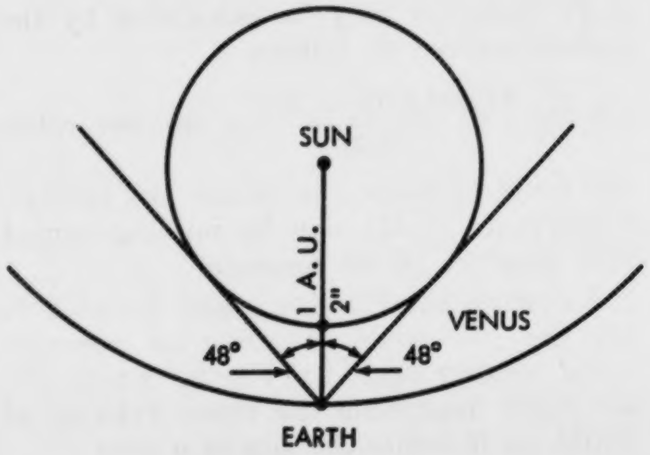


Figure 3–36

As in Figure 3–36 we place a point to represent the sun. From this point we draw a line 2 inches long to represent the Earth's distance from the sun, 1 A.U. On each side of this line we measure an angle of 48°. With the sun as center, we construct a circle that just touches both rays of the angle. Notice that only one circle may be drawn that is tangent to both rays. This circle represents the orbit of Venus. The distance from the sun to the circle can be measured from our scale drawing as 1½ inches. Therefore if 2 inches equals 1 A.U., we are able to give the distance of Venus from the sun as three-fourths of Earth's distance; that is, 0.75 A.U.

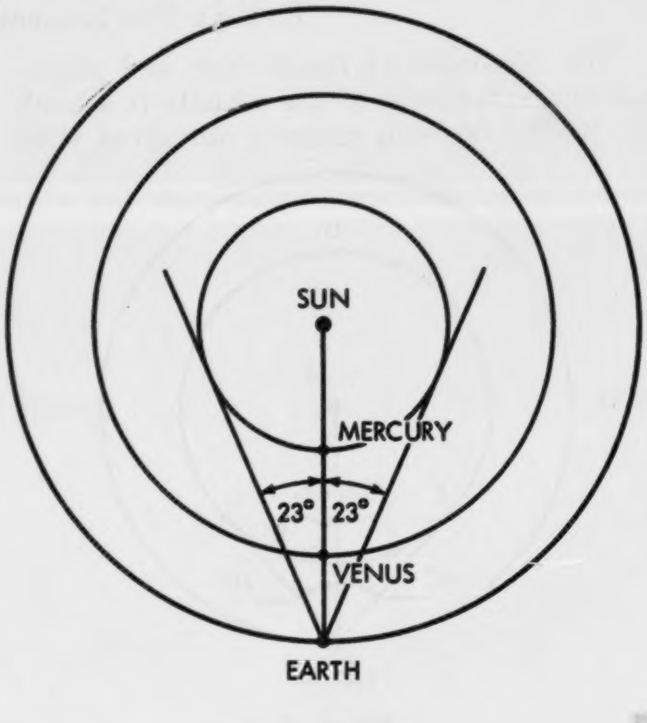


Figure 3–37

We are able to represent the orbit of Mercury in the same manner. The greatest average separation of Mercury from the sun is 23°.

From Figure 3–37 we determine the distance of Mercury from the sun as three-fourths of an inch, thus Mercury's distance from the sun is three-eighths of Earth's distance; that is, 0.375 A.U.

3–9 Exercises—The Inner Planets.

1. How close does Venus come to Earth?
2. What is the greatest distance Venus can be from Earth?
3. What is Mercury's closest distance to Earth? Farthest?
4. If Venus makes one orbit of the sun in 224 days, how many orbits does it make in one year?

3–10 The Outer Planets

To continue our model of the solar system, we must change our scale from 2 inches equal to 1 astronomical unit to 1 inch equals 2 astronomical units. This is necessary due to the very large distances involved. The outer planets are much easier to observe and can be placed in position in their orbits with greater accuracy than the inner planets. Data from the following table will enable us to approximate the position of the outer planets in their orbits.

TABLE 3-3

ORBITS OF PLANETS

	Period	Revolutions of Earth	Separation from sun
Mars	687 Earth days	$1\frac{322}{365}$	101°
Jupiter	11.9 Earth years	11.9	138°
Saturn	29.4 Earth years	29.4	32°

To place Mars and its orbit, we refer to Figure 3–38. The point S represents the position of the sun. A circle with a radius of ½ inch represents the orbit of Earth. A line is drawn from the sun through the circle and out into space. Point E indicates the position of Earth in its orbit. From observations in the sky, we are able to place the planet Mars in the direction of the line. We do not know where, but only that it is opposite the sun as seen from Earth. We know from Table 3–3, that Mars takes 687 days to complete one orbit of the sun, that is to return to the same position in the sky. This is referred to as the "period" of the planet. As Mars orbits the sun, Earth also is orbiting the sun. After 687 days, Earth has completed one revolution and $\frac{322}{365}$ of the second revolution. Earth is now in position E_2. At this time the angle from the sun, Earth and Mars is observed to be 101°, Table 3–3. We draw a line to represent this angle. The point of intersection of this line with the original line of sight to the planet locates the planet Mars at M. Measuring the distance from the sun to M, we estimate the distance to be ¾ of an inch. A circle with a radius of ¾ inch would approximate the orbit of Mars. Mars' distance from the sun is 1.5 times the Earth's distance; that is, 1.5 A.U.

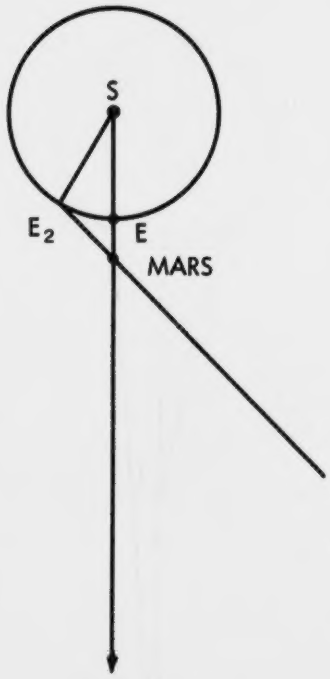


Figure 3–38

We continue as in Figure 3–39 to plot the positions of the planets, Jupiter and Saturn. From Table 3–3 we obtain the period of Jupiter as 11.9 years. As Jupiter makes one revolution around the sun, the Earth makes eleven full revolutions and 0.9 of the twelfth. After 11.9 years, Earth is at position E_2 and Jupiter has returned to its original position in the sky. From Table 3–3, the observed angle from the

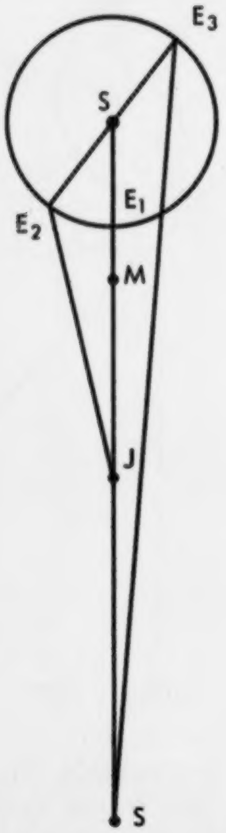


Figure 3–39

sun, Earth, and Jupiter is 138°. Repeating the method of locating Mars, the planet Jupiter is placed at point J. The distance from the sun to Jupiter is estimated to be 2½ inches. A circle with a radius of 2½ inches would approximate the orbit of the planet Jupiter. Jupiter's distance from the sun would be estimated as five times Earth's distance; that is, 5 A.U.

From Table 3–3 we obtain Saturn's period as 29.4 Earth years. During this period of time, Earth would complete 29.4 revolutions. After 29.4 revolutions, Earth would be at E_3 as illustrated in Figure 3–39. From Table 3–3, the angle of the sun, Earth, and Saturn is 32°. This places Saturn at point S. The distance from the sun to Saturn is estimated at 4¾ inches. A circle with a radius of 4¾ inches would approximate the orbit of Saturn. Saturn's distance from the sun would be estimated as 9.5 times Earth's distance; that is, 9.5 A.U.

The outermost 3 planets, Uranus, Neptune. and Pluto, are not visible to the naked eye, but must be observed through a telescope. Their positions may be plotted and added to Figure 3–39. Figure 3–40 combines the inner and outer planets in a single drawing. Figure 3–40 only approximates the distances of the planets from the sun and does not attempt to place them in their orbits.

One system that relates the distances of the planets from the sun is called Bode's Law. The system originated in 1772. Mod-

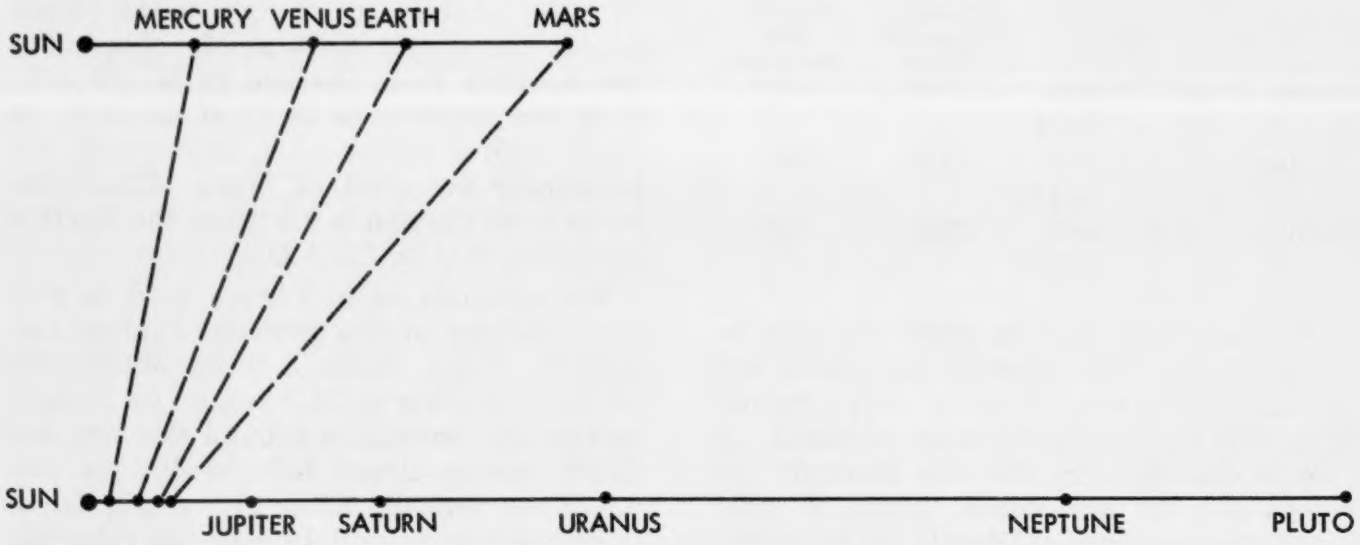


Figure 3–40

ern attempts to explain the relationship according to gravitational theory have failed. It is thought that the relationship is a coincidence. Perhaps as we learn more about the distribution of matter in space, we will be able to explain the relationship. A note of interest is that Bode's Law (Table 3–4) originated in 1772, predicted the presence of a planet orbiting between Mars and Jupiter. In 1801, the first of the asteroids were discovered and the gap was filled.

TABLE 3-4

Bode's Law

Mercury	Venus	Earth	Mars	Asteroids
4	4	4	4	4
0	3	6	12	24
—	—	—	—	—
4	7	10	16	28
0.4	0.7	1	1.6	2.8
(0.39)	(0.72)	(1.00)	(1.52)	(2.65)

Jupiter	Saturn	Uranus	Neptune	Pluto
4	4	4	4	4
48	96	192	384	768
—	—	—	—	—
52	100	196	388	772
5.2	10.0	19.6	38.8	77.2
(5.20)	(9.54)	(19.2)	(30.1)	(39.5)

Table 3–4 lists the planets and asteroids in order from the sun. The number 4 is placed under the name of each member of the solar system. Starting with Mercury, we add 0 to the 4. For Venus we add 3, and double the number for the succeeding members: 2 x 3 = 6 and 6 is added for Earth, 2 x 6 = 12 and 12 is added for Mars, 2 x 12 = 24 and 24 is added for the asteroids, and we continue to Pluto. The resulting numbers are the relative distances of the planets from the sun. If we divide all numbers by 10, we estimate the Earth's distance as 1. If we assign the Earth's distance as 1 A.U., the distances of all the planets are given in astromical units. The figures in parentheses are accepted values in A.U. for the distances to the planets. Notice that the law fails badly in the case of Neptune and Pluto.

Entire books have been devoted to individual planets. We have attempted to obtain an impression of the distances in our solar system. From this point, we leave the planets, and look to the stars.

3–11 Distances to the Stars

An old Chinese proverb stated that the greatest journey starts with a single step. We have taken that first step. From the estimate of the distance to the door in Section 3–1 to the complicated computation of the astronomical unit in Section 3–8, we now reach out to measure the vast dimensions of our universe.

The early philosophers recognized that Earth's motion around the sun should produce a change in the apparent positions of the stars. Because they could not detect a change in the stars' positions, the ancients had to conclude Earth was not moving but fixed. This idea continued until the early 19th Century. Using the most precise instruments and methods available, astronomers finally succeeded in detecting the parallax of some stars. The problem was due to the vast distances involved. Star charts constructed in various countries were similar indicating that the diameter of Earth was much too small to use as a baseline for the measurement of stellar parallax. The mean diameter of Earth's orbit proved to be a satisfactory baseline.

The distance to the nearest stars may be determined by measuring their apparent shift against the more distant background stars. When the Earth is at E_1, Figure 3–41, a photograph is taken of nearby star S against the background stars. Six months later another photograph is taken of star S. The photographs are then studied under a microscope and the apparent shift of the star, stellar parallax, is determined. One-half of this angle, the angle that would be produced using the radius of the Earth's orbit as 1 A.U. is then referred to as the star's *heliocentric parallax, p.* This would be true if Earth's orbit were circular. However, in actual calculations, astronomers allow for the elliptical shape of

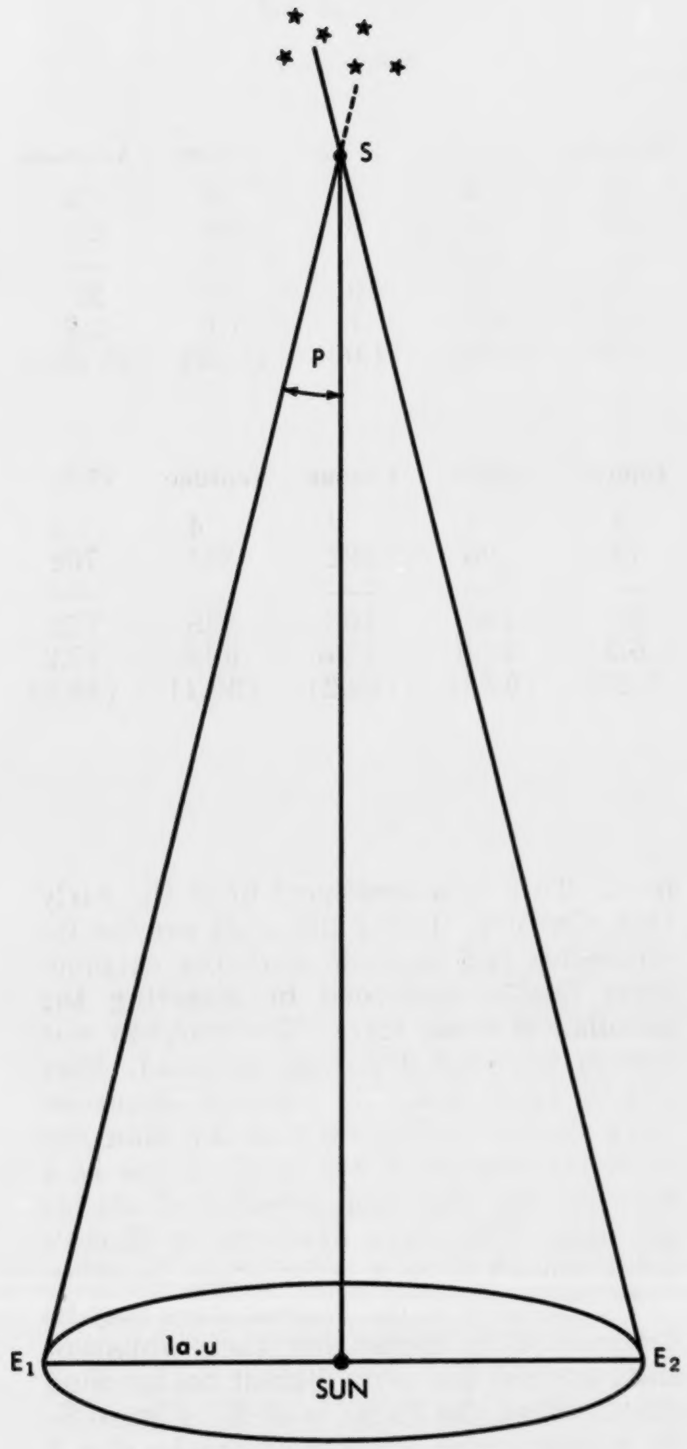


Figure 3–41

Earth's orbit. For our purposes, we shall use a diameter of the circular orbit of Earth with radius 1 A.U. as a baseline. The parallax of a star is extremely small, the largest known stellar parallax is about 0.75 seconds. If a star had a parallax of 1 second of arc, it would be at a distance of 206,265 A.U. Remember that the smaller the parallax the more distant the star. The closest star that can be observed by the unaided eye is Alpha Centauri. It has a parallax of 0.75 seconds; its distance is $\frac{206{,}265}{0.75}$; that is about 275,000 A.U. Again we have a problem of units. As in measuring the distance to the door in inches or the distance to the sun in miles, the large numbers are confusing and somewhat meaningless.

Astronomers have invented two units to measure stellar distances, the parsec and the light year. The parsec is the more convenient but understanding it requires some thought; 1 *parsec* is the distance to a star that shows a parallax of 1″. A distance D in parsecs is given by the formula

$$D = \frac{1}{p}$$

where p is the parallax in seconds. A star that would show a parallax of 1 second would be at a distance of 1 parsec or 206,265 A.U. The sun is the only star that is closer than 1 parsec; that is, the only star that has a parallax greater than 1″. Alpha Centauri's distance in parsecs could be calculated as follows:

$$D = \frac{1}{0.75} = 1.33 \text{ parsecs}$$

One *light year* is the distance that light travelling at 186,000 miles per sec. will travel in one year. *This is not a unit of time.* By simple multiplication, we could determine that 1 parsec is equal to about 3.26 light years. Thus a star's distance in light years may be found by dividing 3.26 by the parallax in seconds of the star. Alpha Centauri's distance in light years is $\frac{3.26}{0.75}$; that is, about 4.3 light years.

We are able to relate the A.U., parsec, and light year as follows:

1 light year ≈ 63,000 A.U.
1 parsec ≈ 3.26 light years

Because of the problems in accurately measuring star images on a photograph, we are able to measure the parallax only for stars up to a distance of about 100 parsecs; that is, about 300 light years. Stars at this distance would have a parallax of about 0.01 seconds. For stars at a greater distance, the method of parallax is subject to an error of greater than 50%. We have been able to acceptably measure the parallax of about 6,000 stars. Some of the better known stars, their parallax, and distance in parsecs and light years are indicated in Table 3–5.

TABLE 3-5

Star	Constellation	Parallax (sec.)	Distance (parsecs	Distance (light years)
Alpha Centauri	Centaur	0.75	1.32	4.3
Sirius	Canis Major	0.37	2.66	8.6
Procyon	Canis Minor	0.28	3.47	11.3
Altair	Aquila	0.19	5.05	16.4
Vega	Lyra	0.12	8.14	26.5
Pollux	Gemini	0.093	10.8	32.2
Arcturus	Bootes	0.090	11.1	36.1
Capella	Auriga	0.07	13.7	44.6
Aldebaran	Taurus	0.04	20.8	67.8
Regulus	Leo	0.03	26.6	83.4

To determine the distance to stars and galaxies more distant than 100 parsecs, we must turn to indirect methods, and therefore to the classification of stars.

3–11 Exercises—Distance to the Stars

Complete the array as far as you can.

	Parallax (seconds)	Parsecs	Light years	A.U.
1.	0.4	____	____	____
2.		55	____	____
3.	____	____	3.74	____
4.	____	____	____	4,221,000
5.	____	45	____	____

3–12 Magnitude and Brightness

When we look upon the starry sky, we see but a few of the brightest stars. Thousands of stars are lost to view in the reflected glare of the light from our major cities and in smoke and dust.

The ancients looked upon a myriad of stars in clear and moonless skies. They saw stars of varying brightness and color and decided to group them according to apparent brightness. The twenty brightest stars of the sky were called stars of first magnitude. The word magnitude does not refer to the *size* of the star but to its *apparent* brightness. Other stars were called 2nd, 3rd, 4th, 5th and 6th magnitude according to their apparent brightness. The 6th magnitude stars are the faintest stars visible to the naked eye under the most favorable conditions. Astronomers were not satisfied with this qualitative scheme but decided to measure the brightness of stars very precisely. They took the average brightness of about the twenty brightest stars and called this brightness 1st magnitude. This left stars that were *brighter than 1st magnitude.* These stars were assigned magnitudes such as 0, —1, and —2. The brightest star of the night time sky, Sirius, was assigned a magnitude of —1.6. Also with the invention of the telescope, stars were discovered that were not visible to the naked eye. These stars, fainter than 6th magnitude, were assigned magnitudes of 7, 8, 9, 10 and so on. On the magnitude scale, the brightest objects have the lowest magnitudes and the faintest objects have the highest magnitudes.

The 200 inch telescope at Mt. Palomar will reveal to the eye stars of magnitude 21 and will photograph stars as faint as magnitude 23. The zero and negative magnitudes are necessary to describe the brightness of objects such as the sun, full moon, some planets and certain stars which are all brighter than 1st magnitude.

Astronomers found that stars of different apparent magnitudes varied in the amount of light emitted. They determined that the ratio of brightness between two

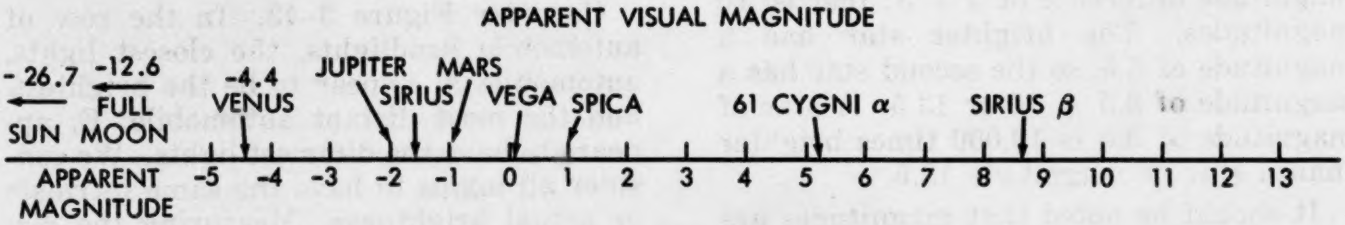


Figure 3–42

stars of successive integral magnitudes was about 2.5. This means two stars differing by 1 magnitude, differ in brightness 2.5 times. A relationship of the brightness and magnitudes of stars was worked out and an approximation is given in Table 3–6.

TABLE 3-6

Difference in Magnitude	Ratio of Brightness
0.00	1.0
0.5	1.6
1.0	2.5
1.5	4.0
2.0	6.3
2.5	10.0
3.0	16.0
3.5	25.0
4.0	40.0
4.5	63.0
5.0	100.0

We can see from Table 3–6 that if two stars vary 1 magnitude, their apparent brightness varies 2.5 times. Stars with 5 magnitudes difference, vary in brightness by a factor of 100.

Example: Find the difference in the brightness of two stars of magnitude 12 and magnitude 8.

The difference in magnitude is 12 — 8; that is, 4. From Table 3–6, a magnitude difference of 4 gives a ratio of brightness of 40. An 8th magnitude star is 40 times brighter than a 12th magniture star.

Example: A star of magnitude 3.5 is 10,000 times brighter than another star. What is the magnitude of the second star?

The brightness ratio of 10,000 may be written as 100 x 100. A brightness difference of 100 is equal to a magnitude difference of 5, Table 3–6. Therefore a brightness difference of 100 x 100 is equal to a magnitude difference of 5 + 5; that is, 10 magnitudes. The brighter star had a magnitude of 3.5, so the second star has a magnitude of 3.5 + 10 or 13.5. A star of magnitude of 3.5 is 10,000 times brighter than a star of magnitude 13.5.

It should be noted that magnitudes are added; brightnesses are multiplied. Magnitude is not proportional to brightness. Although the above approximations are sufficient for our purposes, astronomers are able to determine magnitudes to the nearest thousandth by precise photoelectric methods.

3–12 Exercises—Magnitude and Brightness

1. Star A is of magnitude 4.5 and is 100 times brighter than Star B. What is the magnitude of Star B?
2. A star has a magnitude —1.5 and is 40 times brighter than a second star. What is the magnitude of the second star?
3. The sun has a magnitude —26.7 and the full moon has a magnitude of —12.6. How much brighter is the sun than the full moon?
4. The full moon appears 17,000 times brighter than the planet Jupiter. What is the magnitude of Jupiter?
5. How much brighter is the star Sirius, —1.6 magnitude, than the faintest star visible to the human eye, 6th magnitude?

3–13 Apparent and Absolute Magnitude

In Section 3–12, we referred to the brightness and magnitude of stars and planets as they appear to us in the sky. This is the apparent brightness and apparent magnitude of the object. This tells us little of the actual or intrinsic brightness. Certainly, it is correct to say that the sun *appears* brighter than the star Sirius. Is it correct to say that the sun is actually brighter than Sirius? The sun is at a distance of 1 A.U. from the Earth, whereas Sirius is at a distance of 8.5 light years; that is, 535,500 A.U. How bright would the sun appear if it were moved 500,000 times farther away? Not very bright! The brightness of a light source depends in part upon its distance.

Consider Figure 3–43. In the row of automobile headlights, the closest lights, automobile A, appear to be the brightest, and the most distant automobile, E, appears to have the dimmest lights. We consider all lamps to have the same intrinsic or actual brightness. Measuring the distance to automobile A, we call this dis-

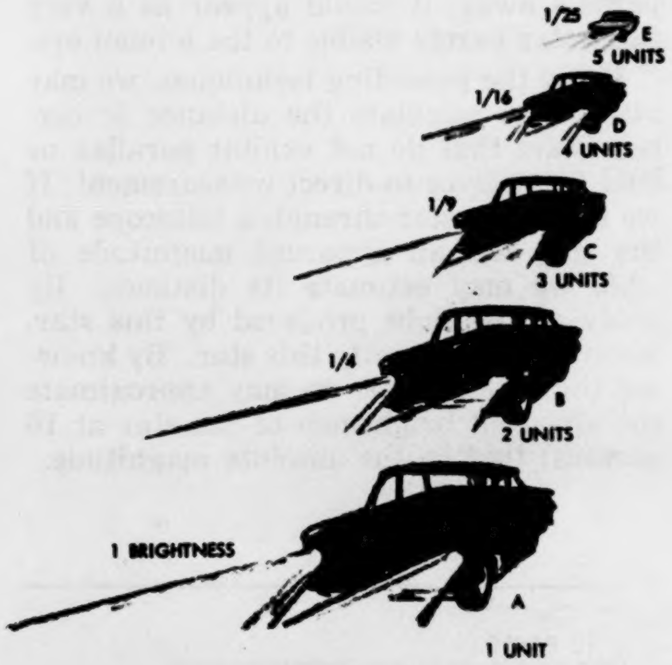


Figure 3–43

tance 1 unit. The amount of light from the headlights of automobile A at a distance of 1 unit may be taken as 1 unit of brightness. We measure the distance to automobile B and find its distance to be 2 units. The light from B is *not* ½ as bright as A but ¼ the brightness of A. An automobile headlight at a distance of 2 units would have a brightness of ¼. Automobile C is at a distance of 3 units and its brightness is ⅑ that of automobile A. We could construct a table of distance and brightness as follows:

Automobile	Distance	Brightness
A	1 unit	1 unit
B	2	¼
C	3	⅑
D	4	1/16
E	5	1/25

In general, the apparent brightness, m, of a light source decreases inversely with the square of the distance. This is true for the light of stars.

If we return to our problem of the sun, we could determine its brightness at a distance of 500,000 A.U. If the sun's distance were increased 500,000 times and placed next to Sirius, the sun would not appear to be $\frac{1}{500.000}$ as bright, but it would appear $\frac{1}{250.000.000.000}$ as bright as it did at a distance of 1 A.U.

Example: If the sun's apparent magnitude at 1 A.U. is −26.7, what is its apparent magnitude at 500,000 A.U. ?

The sun appears $\frac{1}{250.000.000.000}$ as bright; 250,000,000,000 may be factored as 100 x 100 x 100 x 100 x 100 x 25. This is equal to a change in magnitude of 5 + 5 + 5 + 5 + 5 + 3.5; that is, 28.5. The sun's apparent magnitude at 1 A.U. is −26.7, its apparent magnitude at 500,000 A.U. is −26.7 + 28.5; that is, 1.8.

Example: If at the distance of 500,000 A.U., the sun had an apparent magnitude of 1.8 and Sirius has an apparent magnitude of −1.6, how many times brighter would Sirius be than the sun?

Sun 1.8, Sirius −1.6; the difference in magnitude would be 3.4. A difference in 3.4 magnitudes implies a difference in brightness of nearly 25. Sirius would appear nearly 25 times brighter than the sun!

Realizing this basic problem, astronomers proposed a system to compare the actual or intrinsic brightness of stars. The absolute magnitude of a star is the apparent brightness that a star would have at a standard distance of 10 parsec; that is, 32.6 light years. When we refer to absolute magnitude, we are "moving back" nearby stars and "moving up" distant stars to 10 parsecs. If all stars were in a row at equal distances from the Earth, we could compare their actual brightnesses.

Example: What is the absolute magnitude of a star with an apparent magnitude of 1.5 at a distance of 2 parsecs?

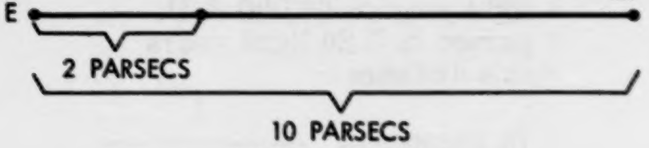


Figure 3–44

The distance to the star would increase from 2 parsecs to 10 parsecs; that is, 5 times. Increasing the distance 5 times, decreases the brightness 25 times. The magnitude changes by a factor of 3.5. If the star had an apparent magnitude of 1.5 at 2 parsecs, its absolute magnitude at 10 parsecs would be 1.5 + 3.5; that is, 5.

Example: A star with apparent magnitude of −1.2 exhibits a parallax of 0.4″ What is its absolute magnitude?

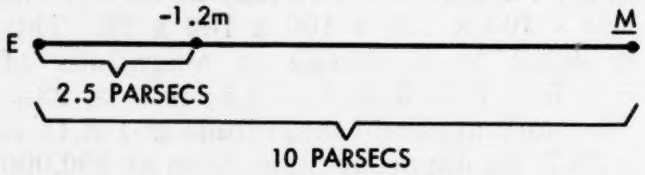


Figure 3–45

$$\text{Distance in parsecs} = \frac{1}{p} = \frac{1}{0.4} = 2.5.$$

The star's distance is increased from 2.5 to 10 parsecs; that is, by a factor of 4. This decreases the brightness by 16. A decrease in brightness of 16, increases the magnitude by 3; −1.2 + 3 = 1.8. A star of −1.2 apparent magnitude with a parallax of 0.4″ has an absolute magnitude of 1.8.

Those students who are familiar with logarithms may calculate the absolute magnitude of a star, $\underline{M}$, as follows:

$$\underline{M} = m + 5 - 5 \log d$$

where m is the apparent magnitude and d is the distance to the star in parsecs.

Example: What is the absolute magnitude of a star at a distance of 8.3 parsecs with an apparent magnitude of 2.6?

$$\begin{aligned} \underline{M} &= m + 5 - 5 \log d \\ &= 2.6 + 5 - 5 \log 8.3 \\ &\approx 7.6 - (5 \times 0.9191) \\ &\approx 7.6 - 4.6 \\ \underline{M} &\approx 3.0 \end{aligned}$$

Example: Calculate the absolute magnitude of the sun. Apparent magnitude −26.7 at a distance of 1 A.U.

1 light year ≈ 63,000 A.U.
1 parsec ≈ 3.26 light years

$$\text{Sun's distance in parsecs} = \frac{1}{3.26 \times 63{,}000} \approx 0.0000048$$

$$\begin{aligned} \underline{M} &= m + 5 - 5 \log d \\ &= -26.7 + 5 - 5 \log 0.0000048 \\ &\approx -21.7 - (5 \times -5.32) \\ &\approx -21.7 + 26.6 \\ \underline{M} &\approx 4.9 \end{aligned}$$

The sun would have an absolute magnitude of about 4.9. If the sun were 10 parsecs away, it would appear as a very faint star barely visible to the human eye.

Using the preceding techniques, we may attempt to calculate the distance to certain stars that do not exhibit parallax or lend themselves to direct measurement. If we observe a star through a telescope and the star has an apparent magnitude of 12.6, we may estimate its distance. By studying the light produced by this star, we are able to classify this star. By knowing the type of star, we may approximate the apparent brightness of the star at 10 parsecs; that is, the absolute magnitude.

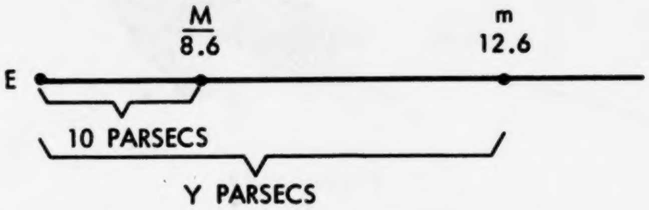


Figure 3–46

Suppose that a star has an absolute magnitude of 8.6. An increase of 4 magnitudes, decreases the brightness by a factor of 40. A decrease in brightness by 40, increases the distance by the $\sqrt{40}$; (that is, about 6.3 times). If a star had an absolute magnitude of 8.6, at 10 parsecs it would have an apparent magnitude of 12.6 at a distance of about 63 parsecs. Using the log formula, the distance may be computed as follows:

$$\begin{aligned} \underline{M} &= m + 5 - 5 \log d \\ \underline{8.6} &= 12.6 + 5 - 5 \log d \\ 5 \log d &= 9.0 \\ \log d &= 1.8 \\ d &= 63.0 \text{ parsecs} \end{aligned}$$

This indirect method provides an approximation for the distance. The method depends upon the astronomer's ability to identify the type of star and to estimate its intrinsic brightness. This enables the star to be assigned an absolute magnitude. We arrive at the question of how stars are classified and the absolute magnitude estimated. For insight into this problem, we turn to Section 3–14 on the classification of stars.

3–13 Exercises Apparent and Absolute Magnitude

1. (a) Complete the following table as far as you can:

Star	Distance (parsecs)	Apparent Magnitude	Absolute Magnitude
A	10	2	
B	100	4	
C	40	0	

(b) List the above stars in order of decreasing apparent magnitude (brightest first).

(c) List the above stars in order of decreasing absolute magnitude.

2. Tabulate the answers to the following problems giving M, *m, distance in parsecs, light years,* and *parallax.* Give the stars brightness compared to the sun. The sun's absolute magnitude is about 4.9.

(a). Star A with apparent magnitude (m) — 1.8, D = 3.57

(b). Star B with m = —0.3, parallax 0.03″.

(c). Star C with m = 7.9, D = 10 parsecs

(d). Star D with M = 4.37, parallax 0.45″

3–14 Classification of Stars

When we observe the stars, we can only study the light of the star. Scientists are not directly able to measure, weigh, or to take the temperature of a star. By combining the study of observed light and laboratory analysis, astronomers are able to make predictions as to the chemical and physical properties of the stars.

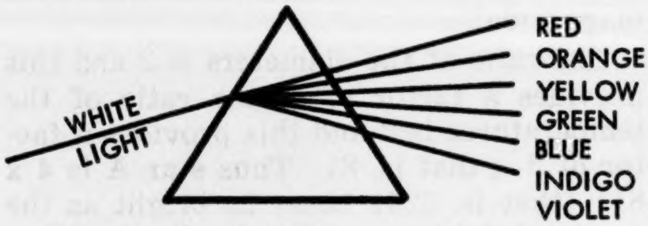


Figure 3–47

When white light is passed through a prism, it is separated into its component wavelengths which we identify as colors.

The short wavelengths produce the sensation in our minds of the colors we call violet and blue. The longer wavelengths produce the red colors of the visible spectrum. Invisible to the human eye are the very short ultra violet wavelengths, and the very long infrared wavelengths. To detect these wavelengths, scientists must turn to specialized instruments.

There are three types of spectra. A spectrum is the band of light produced when light is separated into its component parts. To separate the light into its spectrum, astronomers use a diffraction grating instead of a prism. The diffraction grating is a special metal or glass plate with extremely fine grooves, about 30,000 per inch.

The first type of spectrum is produced from glowing solids or gases at high pressure. An example of this is an incandescent light bulb. In this *continuous spectra* colors blend without interruption.

A gas heated under low pressure produces a *bright line* (or *emission*) *spectrum.* This second type spectrum consists of specific bright lines depending upon the chemical composition of the gas.

If the light from a glowing gas under high pressure, continuous spectrum, is permitted to pass through a cooler gas at low pressure, the spectrum will be continuous missing only certain dark lines. Each dark line will be in the exact position of the bright-lines spectrum produced by the cooler gas if it were heated to glowing. The *dark line spectrum* is also called an *absorption spectrum.* This is the third type of spectrum produced by most stars.

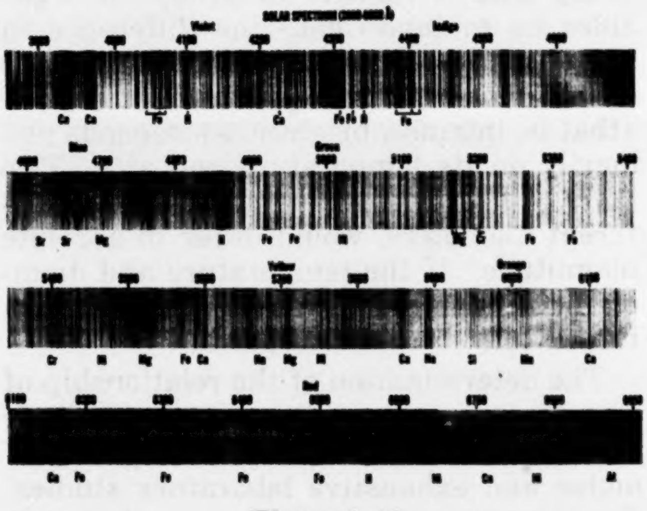


Figure 3–48

"Photo from the Mt. Wilson and Palomar Observatories"

Figure 3–48 shows part of the absorption spectrum of the sun. This mass of dark lines is representative of many vaporized elements found in the outer layers of the sun. Comparing these dark lines in the spectrum of the sun to known laboratory spectra, the dark lines can be identified as being produced by certain chemical elements found in outer layers of the sun. Using this method, more than 60 of the 92 naturally occurring elements of Earth have been identified in the sun. It is thought that nearly all the natural elements of Earth are found in the sun and are waiting to be identified.

By comparing the spectra of various stars to known laboratory spectra, elements may be identified in the stars. This leads to the classification of stars. Stars are classified primarily according to temperature. Stars that have the strongest portion of their continuous spectrum in the blue region are found to be of very high temperature. Stars with the strongest portion in the yellow are average temperature and the coolest stars have the strongest portion of the spectrum in the red and orange.

Star Color	**Star Temperature**
Blue white	20,000—40,000°F
Yellow	5,000—8,000
Red and orange	4,000—5,000

These are general classifications, and individual investigations of stars can approximate their individual temperatures. Being able to classify or group stars enables us to investigate the difference in the absolute magnitude of various types of stars. A star's absolute magnitude (that is, intrinsic brightness) depends primarily on its temperature and size. Two stars of the same temperature, but of different diameters, would differ in absolute magnitude. If the temperature and diameter of a star is known, its absolute magnitude may be approximated.

The determination of the relationship of size, temperature, and absolute magnitude for stars has resulted from complex formulae and exhaustive laboratory studies. For our purposes we will assume that each unit area of a star at a given temperature radiates equal amounts of energy. Then two stars of the same size and temperature will radiate equal amounts of energy.

The brightness of a star varies as the fourth power of the temperature, in absolute units (Kelvin scale).

Example: Star A and star B are the same size. The temperature of star A is twice the temperature of star B. Star B has an absolute magnitude of 5.2. What is the absolute magnitude of star A?

The ratio of the temperatures is 2; the ratio of the brightnesses is 2^4; that is, 2 x 2 x 2 x 2. Star A is 16 times brighter than B. A brightness ratio of 16 implies a difference in magnitude of 3. 5.2 — 3 = 2.2. The absolute magnitude of star A is 2.2.

Stars of the same temperature, but of different sizes may vary in brightness. The brightness between two stars varies as the surfaces of the stars and thus as the squares of their radii.

Example: Star C and star D are of the same temperature but vary in size. Star C has a diameter of 800,000 miles and star D has a diameter of 1,600,000 miles. How much brighter is star D than star C? The ratio of the radii is 2 to 1; the ratio of their brightnesses is 4 to 1. Star D is 4 times as bright as star C. Notice that there would be a difference of about 2.6 magnitudes.

The two comparisons that we have considered may both be used for the same stars.

Example: Star A's diameter is twice that of the sun and its temperature is 3 times as great. What is star A's absolute magnitude?

The ratio of the diameters is 2 and this provides a factor of 4; the ratio of the temperatures is 3 and this provides a factor of 3 ; that is, 81. Thus star A is 4 x 81; (that is, 324) times as bright as the sun. A brightness of 324 implies a difference in magnitude of about 6.5. The sun's absolute magnitude is 4.9; 4.9 — 6.5 = —1.6. Star A's absolute magnitude is about —1.6.

Stars vary greatly in temperature and diameter and therefore differ in absolute magnitude. The stars from Table 3–5 are listed in Table 3–7 according to their absolute magnitude. Compare the stars with respect to their absolute magnitudes, temperatures, and sizes. Then compare the stars in absolute and apparent magnitudes and distances. Note that the radii are expressed on a scale such that the sun has radius of 1 unit.

TABLE 3-7

Star	Absolute Magnitude	Temperature °K	Radius	Apparent Magnitude
Aldebaran	−0.8	3,600	53.0	0.78
Regulus	−0.7	13,000	3.5	1.33
Arcturus	−0.2	4,000	29.0	−0.06
Capella	+0.2	5,200	11.0	0.90
Vega	+0.4	10,700	2.5	0.00
Sirius	+1.4	10,000	1.8	−1.44
Altair	+2.2	8,000	1.7	0.76
Procyon	+2.6	6,500	2.0	0.36
Alpha Centauri	+4.6	5,800	1.0	0.30
Sun	+4.9	5,500	1.0	−26.7

By classifying stars, we are able to estimate their temperatures and sizes. From the temperature and size of a star we approximate its absolute magnitude. By the comparison of the apparent and absolute magnitudes, we arrive at an estimation of the star's distance.

The preceding discussions of determining distance, magnitude, brightness and star classification is a simplification of one method. Through the years, astronomers have developed many instruments. These new instruments have enabled man to determine distances in numerous new and exciting ways.

3-14 Exercises—Classification of Stars

1. A star which increases and decreases in diameter at regular intervals of time is called a pulsating star. A "pulsating" star varies in diameter from a minimum value of 25 times the sun's diameter to a maximum of 36 times the sun's diameter. How many times brighter is the star at maximum size than at minimum size? How many magnitudes difference does this represent?
2. A star's temperature is 2.5 times the sun's and its diameter is 32 times the sun's diameter. How many times brighter is this star than the sun?
3. A star's absolute magnitude is -1.2. Its temperature is 2 times the sun's temperature. What is its diameter in terms of the sun's?
4. A star has an apparent magnitude of 7.6. It is 2 times the sun's diameter but only $\frac{1}{4}$ of the sun's temperature. What is its distance in light years?
   Note: Problems of this type are applicable to computer programming. See Section 6–5, Exercise 3.

3–15 Frontiers of the Universe

Specialized methods of determining distance have been discussed in the last few sections. These methods have enabled man to make more varied and precise calculations.

Originally the term "fixed" star was used to separate the planets from the stars. The planets were considered "wanderers." After careful study with modern telescopes, the individual star motions were detected. From the motion of a star through space, we are able to calculate the star's space velocity, its motion relative to the sun. This enables us to understand better our star, the sun, and our galaxy, the Milky Way. From the study of the light from a special class of stars, variable or "pulsating", man is able to predict their absolute magnitude.

In 1952 observations with the 200 inch telescope at Mount Palomar literally doubled the size of the visible universe. Prior to this time, astronomers had placed the nearby galaxy, Andromeda, at a distance of 800,000 light years. By studying a class of pulsating stars, Classical Cepheids, in the Andromeda galaxy, astronomers found that the galaxy was much further away than originally thought. Current estimates place the galaxy at about 2,000,000 light years. The outcome of this observation was to place this galaxy and all similar galaxies at twice their original distances. This one discovery for all practical purposes doubled the size of man's known universe.

Man has been able to estimate the distances to far off galaxies and celestial objects. The energy from distant galaxies has its maximum value in the yellow region of the spectrum. Galaxies appear to be receding from the sun and Earth. As the distance to the galaxies increases, the maximum value of energy is shifted toward the red area of the spectrum. This is referred to as the "red shift." The distance and speed of recession of the galaxy is related to the amount of the shift.

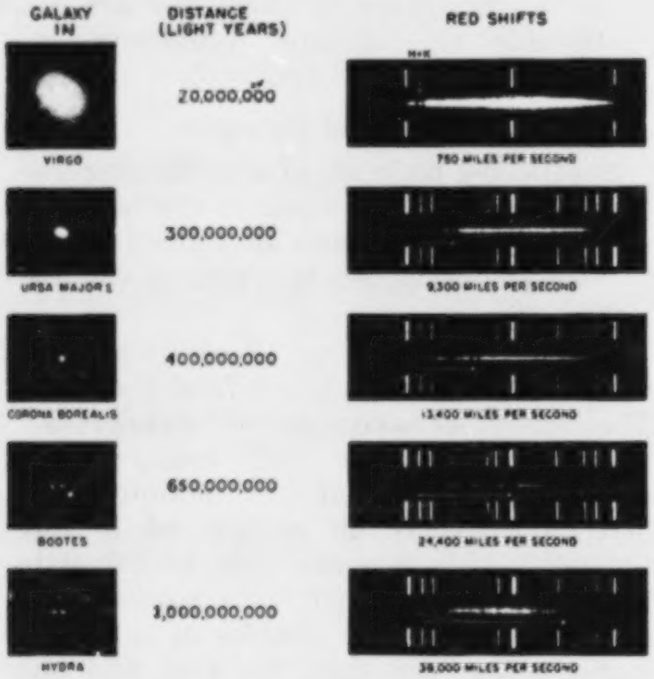

GALAXY IN
DISTANCE (LIGHT YEARS)
RED SHIFTS
H+K
20,000,000
VIRGO
750 MILES PER SECOND
300,000,000
URSA MAJOR I
9,300 MILES PER SECOND
400,000,000
CORONA BOREALIS
13,400 MILES PER SECOND
650,000,000
BOOTES
24,400 MILES PER SECOND
1,000,000,000
HYDRA
38,000 MILES PER SECOND

Figure 3–49

"Photograph from the Mt. Wilson and Palomar Observatories"

Radio astronomy is a new and interesting branch of astronomy. Many remarkable discoveries have taken place through the observations of the giant radio telescopes. Current investigations of the "red shifts" produced by certain radio sources, place these sources over 100,000,000 parsecs from the sun. It is theorized by some that if these sources were only 20 times farther away they would be at the edge of our universe.

"One-hundred million parsecs"—again we have a problem of units! As the parsecs replaced the astronomical unit, the mega-parsec replaces 1,000,000 parsecs. The distances to these strange radio sources would be approximated as 100 megaparsecs. If we stretched our method of determining distance to its present limit, we might place the edge of our universe at 2,000 megaparsecs. The megaparsec is probably the largest unit that will be necessary to describe our universe. However, it should be remembered that to the Greeks the stadia was the largest unit necessary to describe their universe. We may be confident that in the future if the need arises, astronomers will invent a new and larger unit.

We have travelled a great distance since estimating the distance to the door. Anyone care to calculate the number of inches in a megaparsec? You may *now* convert to more familiar units!

Chapter 4

MOTION IN SPACE

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MOTION IN SPACE

You may think that you are "sitting still" as you read this book. Actually you are traveling through space at about 18.5 miles per second as Earth orbits the Sun. In order to be conscious of movement, we need to have a point of reference. We "see" an object as moving when we can compare it to something in a different state of motion.

It was many centuries before man was able to prove that Earth was moving in space. Now we know that our solar system as well as the other bodies in space move in relationship to one another. As we begin a more detailed investigation of space and the mathematical models we may employ to describe it, we will be concerned with three or more dimensions.

The charts and drawings in this chapter are on two-dimensional planes . . . the pages of this book. It will help you to construct three-dimensional models as you go along. Use the charts and suggestions as guides for your constructions. When you have finished see if you cannot better explain three dimensional space to someone else. Then try four dimensions. You may be surprised at how well you will be able to do it.

This chapter is a brief resumé of some of the fundamental knowledge of motion that has been gathered over the ages and codified into two topics: (1) kinematics, the descriptive language of motion; and (2) dynamics, the controlling factors of motion.

4–1 What Is Motion?

The questions a space scientist raises concerning satellites are not unlike the questions you would consider when planning an extensive trip. What is the distance? How long will it take to get there? When is the best time to start so that you arrive at the right time?

Travel in space requires that we find answers to such questions with a high degree of accuracy. Therefore, we must know many things about motion. Studies indicate that there are common patterns to all objects in motion.

On the surface of Earth, an odometer (the mileage meter part of the speedometer) tabulates the distance travelled (Figure 4–1). A navigator uses the coordinates of his points of origin and destination to compute distance. The use of coordinates in measuring distances in space requires the extension of a two-dimensional coordinate system for a plane to a three-dimensional coordinate system.

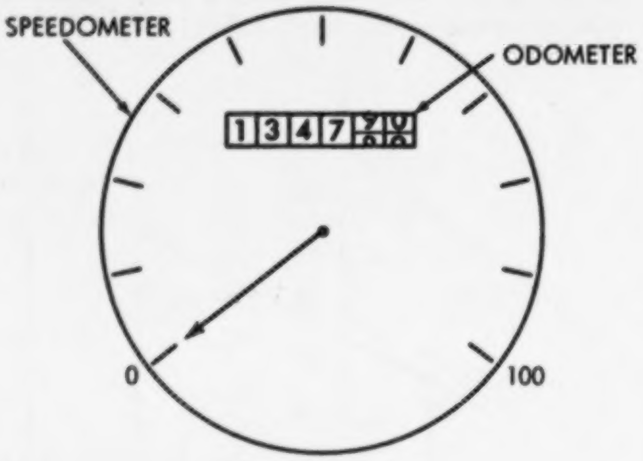


Figure 4–1

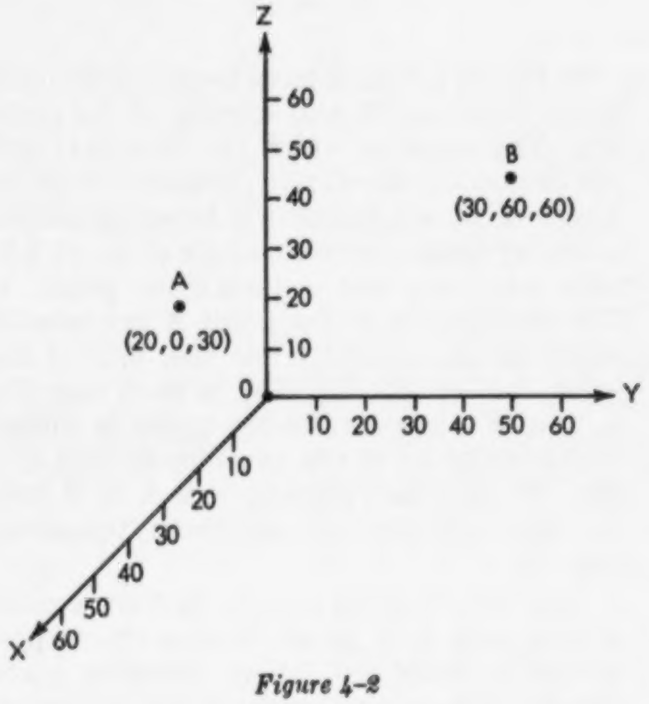


Figure 4–2

In Figure 4–2 can you tell the distance from 0 to A? From A to B? In the type of three-dimensional coordinate system shown in Figure 4–2 there is an origin 0, a y-axis 0Y usually directed toward the right-hand edge of the page, an x-axis 0X directed away from the page (toward you), and a z-axis 0Z directed toward the top of the page. Each axis is perpendicular to the other two. Any two of these axes will determine a plane. Can you find three planes determined by the axes? What angles do these planes (the *coordinate planes*) make with each other?

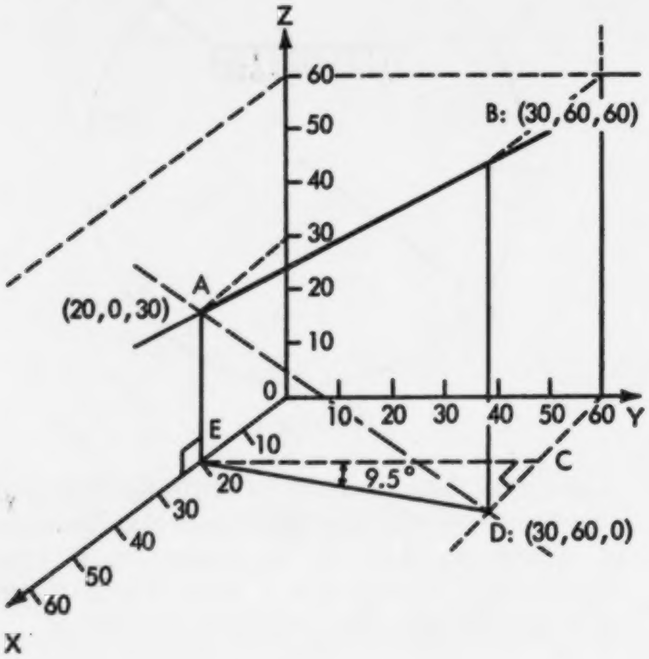


Figure 4-3

In Figure 4–3 lines have been added to aid in the visualization and solution of the problem. The plane in which the lines AD and AB lie is not a coordinate plane and could be described by stating that it is perpendicular to the xy-plane, forms an angle of about 9.5° with yz-plane, and contains the point A. The coordinates of the point A are usually stated as an ordered triple (20, 0, 30); the point B is at (30, 60, 60). In each case the x, y, and z coordinates are given in alphabetical order as in the two-dimensional system. To find the distance from A to B look for right triangles and use the pythagorean theorem.

How long would it take to fly from a point A to a point B in space? Notice that a prediction is asked for. Many scientific space vehicles gather data continuously and transmit information at intervals when they are within range of a receiving station on Earth. The receiving station must be prepared to accept the information at a predetermined time and from a predetermined position (point) in space. The determination of the time and place of the space vehicle requires that the time of flight between two points be predicted accurately. Consider the problem of receiving the TV picture data about Mars transmitted by Mariner IV. Such calculations can be made because the characteristics of the motions of space vehicles are known.

By observing the time it takes an object to travel a known distance a rate may be determined. The *rate* at which an object travels is equal to the distance it travels in a unit time. The selection of time measuring units is arbitrary but seconds or hours are customary. In every day experience the hour is usual. For example, the legal speed limit may be 50 miles per hour.

On the Expressway connecting Baltimore, Maryland and Washington, D. C., there is a measured distance of 5 miles. Suppose you timed your trip over this distance at 5 minutes. Your rate of travel (speed) would be 5 miles in 5 minutes; that is, 1 mile per minute. This rate is often expressed in other ways (Table 4–1). Because of the common usage of miles per hour and feet per second, it is helpful to remember that 88 ft/sec = 60 mi/hr.

Table 4–1

Determination of speed

Distance/time	Rate	Unit time
$\frac{\text{5 miles}}{\text{5 minutes}}$	$1\ \frac{\text{mile}}{\text{minute}}$	1 minute
$\frac{\text{5 miles}}{\text{1/12 hour}}$	$60\ \frac{\text{miles}}{\text{hour}}$	1 hour
$\frac{\text{5 miles} \times 5280\ \frac{\text{feet}}{\text{mile}}}{\text{300 seconds}}$	$88\ \frac{\text{feet}}{\text{second}}$	1 second

The rate r for traveling any distance d in a time t may be found by dividing d (a number of units of lengths) by t (a number of units of time);

$$r = \frac{d}{t} \qquad (1)$$

Notice that for any given period of time the rate is proportional to the distance; $r = (1/t)d$. The rate r is a number of units which we may identify as

$$\frac{\text{unit of length}}{\text{unit of time}}$$

Table 4–2

End of flight	Total distance travelled	Distance during interval			Average speed ft/sec	Change of rate ft/sec²
		from	sec.	to		
0 sec.	0 ft				0	0
1	25	0	1	25 ft	25	
2	100	1	2	75	75	
3	225	2	3	125	125	
4	400	3	4	175	175	
5	625	4	5	225	225	
6	900	5	6	275	275	
7	1225	6	7	325	325	
8	1600	7	8	375	375	
9	2025	8	9	425	425	
10	2500	9	10	475	475	

Measuring the speed of a satellite in a circular orbit is somewhat like driving on a turnpike or throughway. If the measurements were made at several different places the results would be the same. The speed would be uniform. Other types of driving such as on city streets or roads with traffic lights and the motion of the satellite during launching would give various speeds at different times.

Some theoretical measurements of distances during the launching of a model rocket are given in Table 4–2. We shall use three related figures (Figures 4–4, 4–5, and 4–6) to help us understand the various speeds of the rocket.

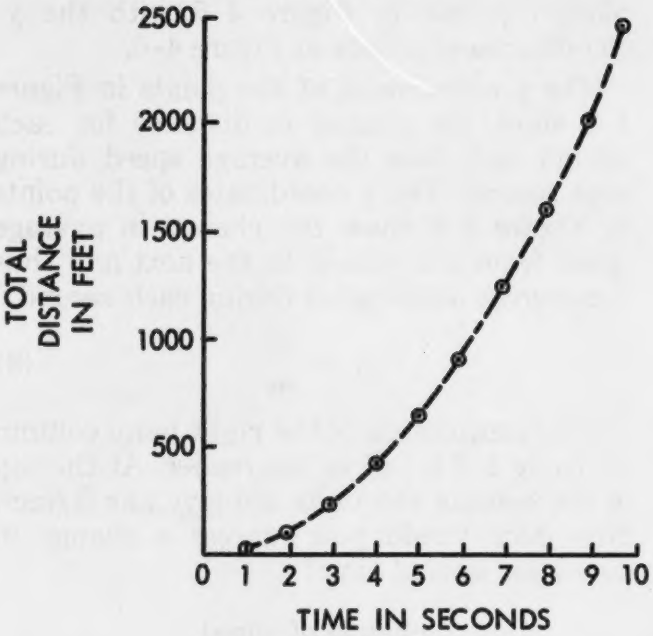


Figure 4–4

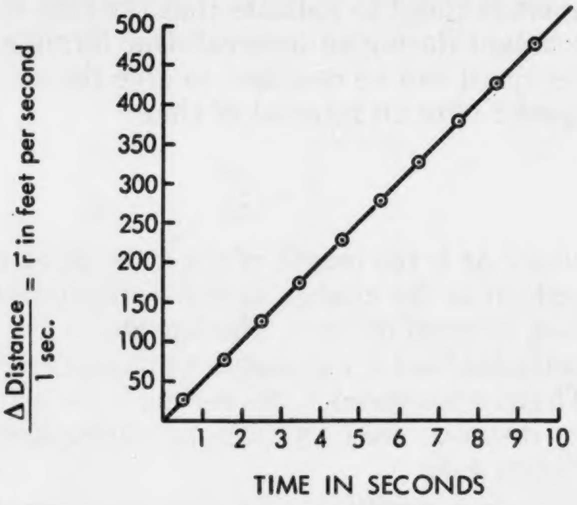


Figure 4–5

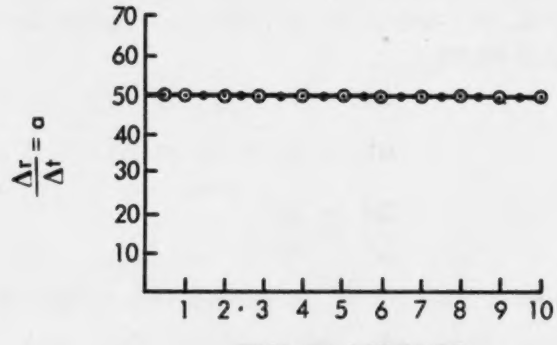


Figure 4–6

The points in Figure 4–4 show the total distance (y-coordinate) travelled in x seconds. Notice that the x-coordinates indicate the ends of the time interval.

The points in Figure 4–5 show the distance (y-coordinate) travelled during each second; each point is plotted over the midpoint of the interval for that second. In Figure 4–4 we are concerned with the total distance traveled up to a specified time; in Figure 4–5 we are concerned with the distance (as determined from Table 4–2) travelled during each second. In Figure 4–5 the y-coordinate of each point is a rate (speed) since it is a measure of distance for a unit of time.

Scientists and mathematicians are careful that their words describe a situation very closely. Examine the data in Table 4–2 and Figure 4–4. During any interval does the rocket have a speed? (Notice the singular, speed) Even during a small interval the speed is changing. The column for speed is headed average speed to indicate that the rate is not constant during an interval. Our formula (1) for speed can be modified to give the *average speed* $\bar{r}$ over an interval of time:

$$\bar{r} = \frac{\Delta d}{\Delta t} \qquad (2)$$

where Δt is the length of the interval of time and Δd is the change in the distance during that interval of time. The bar above the "r" indicates that it represents an average value. The average speed, $\bar{r}$, per second is the change in distance each second as illustrated in Figure 4–5.

Each y-coordinate of a point in Figure 4–5 may be obtained from the coordinates of points in Figure 4–4. Consider any two successive points (x_1, y_1) and (x_2, y_2) in Figure 4–4. We use Δ to indicate a change in value and have

$$\Delta d = y_2 - y_1 = \Delta y$$

$$\Delta t = x_2 - x_1 = \Delta x$$

$$\frac{\Delta d}{\Delta t} = \frac{\Delta y}{\Delta x}$$

The value of $\frac{\Delta y}{\Delta x}$ is called the *slope* of the line through the points (x_1, y_1) and (x_2, y_2).

Figure 4–7 can be considered an enlargement of a small portion of Figure 4–4 to show 2 successive plotted points (x_1, y_1) and (x_2, y_2). Do you see that the point (x_2, y_1) has coordinates $x_1 + \Delta x$ and $y_2 - \Delta y$? The lines joining (x_2, y_1) to the other two points are perpendicular. The triangle having the three points (x_1, y_1), (x_2, y_1), and (x_2, y_2) as vertices is a right triangle; the tangent of the angle at (x_1, y_1) is equal to the slope $\Delta y/\Delta x$ of the line through (x_1, y_1) and (x_2, y_2). You probably realize that unit designations will have to be tagged on to the slope in order to apply the numbers to this problem. Slopes are mathematical concepts which are applicable to many types of problems. Compare the slopes of the lines joining successive plotted points in Figure 4–4 with the y-coordinates of points in Figure 4–5. Compare the slopes of the lines joining successive plotted points in Figure 4–5 with the y-coordinates of points in Figure 4–6.

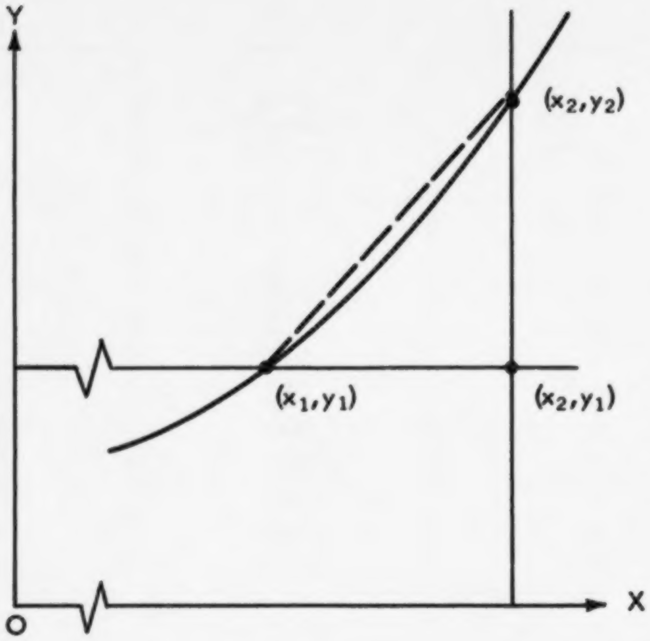


Figure 4–7

The y-coordinates of the points in Figure 4–5 show the change in distance for each second and thus the average speed during each second. The y-coordinates of the points in Figure 4–6 show the change in average speed from one second to the next and thus the *average acceleration* during each second,

$$\bar{a} = \frac{\Delta v}{\Delta t} \qquad (3)$$

The completion of the right hand column of Table 4–2 is left to the reader. At the top of the column the units are given as ft/sec². Now how would you express a change in speed per unit of time?

$$\frac{\text{change of speed}}{\text{unit of time}}$$

Speed is expressed as ft/sec so if the measurement of two speeds are subtracted the units for the change of speed will also be ft/sec. Then change of speed per second can be symbolically written $\frac{\text{ft/sec}}{\text{sec}}$.

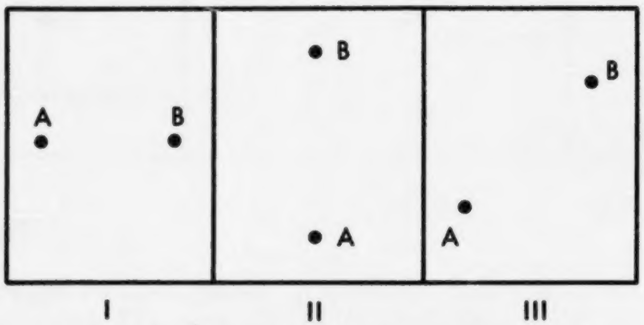
Unit names are not numbers but it is customary in science to write and handle them in a mathematical manner for ease and simplicity of communication. Then $\frac{\text{ft/sec}}{\text{sec}}$ is stated as ft/sec^2 and can be read either as feet per second squared or feet per second per second.

4-1 Exercises What Is Motion?

1. Mariner IV was about 135,000,000 miles from Earth when it was directed by a radio signal from Earth to take pictures of Mars. What interval of time was required for the signal from Earth to reach Mariner IV? The speed of light is about 186,000 mi/sec.
2. Your local newspaper may give you the times when Echo I passes over your locality. This satellite has an approximately circular orbit 1000 miles above Earth and has been timed at 118.3 minutes for one revolution. What is its speed? (Remember that Earth's radius is about 4000 miles.)
3. The Alouette Program is an example of the cooperative effort between governments to expand man's knowledge of space. Canadian scientists designed and built the 320-pound satellite called Alouette; NASA launched it from the Pacific Missile Range. Alouette's orbit is nearly circular at 630 miles altitude. The time of revolution is 105.4 minutes. How does the speed of Alouette compare with that of Echo I? You might want to speculate as to why there is or is not a difference (see Chapter 5).

4-2 Road Maps Without Roads

Have you ever given direction to a traveller as to how to get to a distant town? With route numbers and well marked intersections available, the task is not difficult. Now imagine having to tell a pilot how to get to a distant place. Obviously it can be done even on flights over the ocean where one wave is not distinguishable from another. The pilot would like to know two things: (1) How far is the destination and (2) in what direction is the destination. Information like this which contains a measurement and direction has a special name in mathematics and science. It is called a *vector quantity* and can be represented by a *directed line segment.* (There are some minor differences in the way mathematicians and scientists define and symbolize vectors but the fundamental concepts and operations are the same.)



What is the distance from A to B?

Figure 4-8

In Figure 4-8, how would the vector information be stated? The distance would be easy after selecting the unit length. This value is called the *magnitude* of the vector. In Section I of the Figure most people would say that the direction is East. (Would this do if A and B were locations on a star map?) In II and III you might also give compass directions since most people *refer* to compass directions because of their experience in reading maps.

The word "refer" has particular significance. In addition to seeing the bare details in the picture, you have superimposed—in your imagination—lines which indicate direction. If the units of lengths are included on these superimposed reference lines then the lines are called a *frame of reference.* Figure 4-8 would then appear as in Figure 4-9.

In I of Figure 4-9, B is 400 miles to the East of A. The directed line segment AB is a *vector* and represents the displacement from A to B; that is, $\overrightharpoon{AB}$ has a magnitude of 400 miles and a direction of East. Note the way some mathematicians indicate that a directed line segment from A to B is a vector by a half arrow with the point of the half arrow over B. A single letter ($\overrightharpoon{s}$) could also indicate a vector. A vector has magnitude

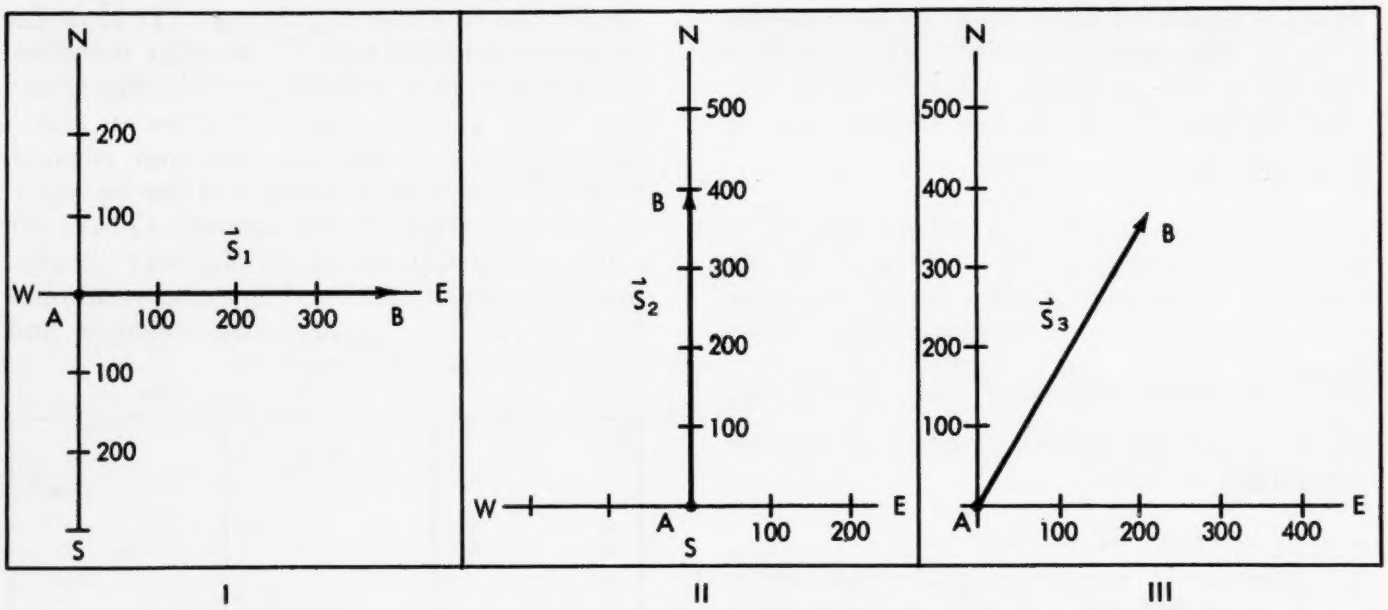


Figure 4–9

and direction (including sense such as East or West along the line). In case III AB has a magnitude of 400 miles and direction of 30° East of North. How would you describe AB in case II?

The three vectors in Figure 4–9 have the same magnitude but not the same direction. Therefore they are three different vectors. For convenience let us refer to them as $\vec{s}_1$, $\vec{s}_2$, and $\vec{s}_3$. Suppose a vehicle started from a place of origin and moved as directed by $\vec{s}_1$ and upon completing the displacement $\vec{s}_1$ moved from that place according to $\vec{s}_2$, and then continued on as directed by $\vec{s}_3$. How far would the vehicle be displaced from the first starting point? Although you are solving this on a flat piece of paper this problem is similar to the problem of a space scientist in knowing where his space vehicle is at all times.

Distance, a measurement of length, is a scalar quantity. Distance with a specific direction associated with it can be represented by a vector. Speed (a rate) is a scalar quantity but speed with a specific direction associated with it is a vector quantity, and is called *velocity*. Since a vector is a directed line segment, the previous statement means that in a frame of reference a velocity can be represented by a directed line segment as in Figure 4–10.

It is important that *scalars* (measures without direction) and vectors be readily identified because there is a difference in the way that mathematical operations apply to them.

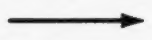
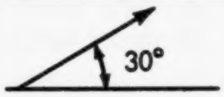
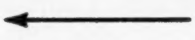
MAGNITUDE DIRECTION	VECTOR
45 mi/hr N	SCALE: 1/8 in = 25 mi/hr.
120 mi/hr E	
140 mi/hr 30° N of E	
165 mi/hr W	

Figure 4–10

In this section you have found that a displacement can be represented by a vector; also the result obtained by two given displacements one after the other is a displacement and is represented by the vector that is the *vector sum* of the vectors for the given displacements.

4–2 Exercises Road Maps Without Roads

Make vector representations of the following situations. For each flight determine the distance of the terminal point from the starting point.

1. Pilot A flys 75 miles due East and then 75 miles due East.

2. Pilot B flys 75 miles due East and then 75 miles NE.
3. Pilot C flys 75 miles due East and then 75 miles North.
4. Pilot D flys 75 miles due East and then 75 miles NW.
5. Pilot E flys 75 miles due East and then 75 miles West.
6. In Exercises 1 through 5 notice how the values of the magnitudes of the sum of two vectors changes as the angle between the vectors changes. How large and how small can the magnitude of $\vec{a} + \vec{b}$ become for given vectors of equal magnitudes?

4–3 Velocity Vectors

A displacement has magnitude (distance) and direction. Therefore we represent displacements by vectors, such as $\overrightarrow{AB}$. The result of two successive displacements is a vector quantity; any change in (difference of) displacements is a vector quantity.

The rate (of speed) at which an object travels has been defined as the change in distance per unit time. Thus rate is represented by a scalar; it has magnitude but not direction. We may associate a direction with a rate by considering displacement instead of distance. The rate with its direction is the *velocity* of the object; we write

$$\vec{v} = \frac{\overrightarrow{\Delta s}}{\Delta t}$$

where $\vec{v}$ and $\overrightarrow{\Delta s}$ are vectors and Δt is a scalar since t is a scalar.

Consider a rocket with a velocity of 1000 ft/sec and going straight away from the Earth (0° to the zenith). Then the velocity is changed so that the direction of flight is at 10° to the zenith. The velocity has changed because one of the characteristics (direction) of the velocity has changed. This is pictured by vector representation in Figure 4–11. A change in velocity is a vector quantity and is found by subtracting the first vector from the second. How does vector subtraction differ from vector addition?

The successful insertion of a satellite into orbit depends on the ability of the launching vehicle to produce the exact terminal velocity (that is, both the proper speed and direction of the satellite) that is, necessary at the instant the satellite is separated from the last stage rocket or its engines are shut off.

4–3 Exercises Velocity Vectors

1. At a certain instant in its upward flight a sounding rocket has a velocity of 2000 ft/sec at 0° to the azimuth. At this instant upper air currents from West to East give the rocket a velocity of 200 ft/sec in the direction of the current. What is the true velocity of the rocket?
2. At a point in its path a rocket which is falling to Earth has a speed of 3000 ft/sec and the flight path makes an angle of 60° with a horizontal line. What are the horizontal and vertical (x and y) components of the velocity?

4–4 Acceleration Vectors

In future sections it will be noted that the rate of change of velocity is an important quantity in analyzing the ways motion may be altered. If you apply previous techniques for obtaining rates you may see how the rate of change of velocity is obtained. We may find $\overrightarrow{\Delta v}$ by using vectors. Then all that needs to be done is to time the interval over which the change took place and we will have $\overrightarrow{\Delta v}/\Delta t$. Dividing a vector $\overrightarrow{\Delta v}$ by a number

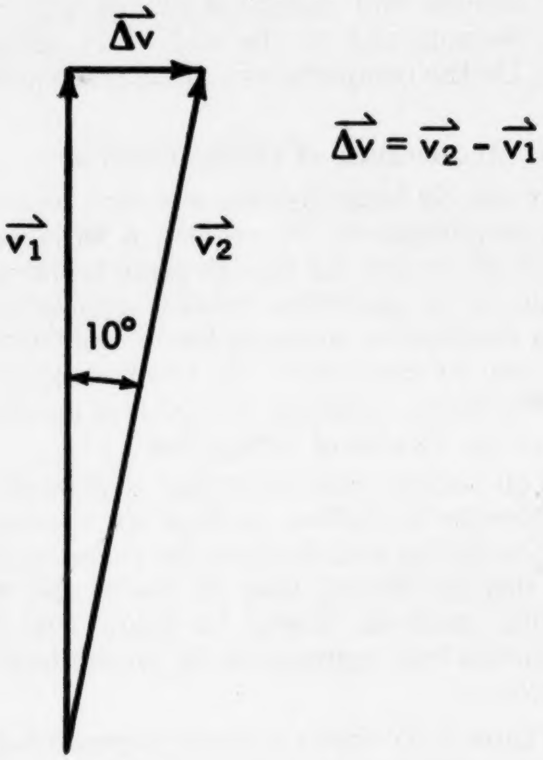

$\overrightarrow{\Delta v}$
$\overrightarrow{\Delta v} = \overrightarrow{v_2} - \overrightarrow{v_1}$
$\overrightarrow{v_1}$
$\overrightarrow{v_2}$
10°

Figure 4–11

Δt is permissible and results in a vector. Then write

$$\vec{a} = \frac{\overrightarrow{\Delta v}}{\Delta t}$$

and use the half arrow to show whether we are thinking of acceleration as a vector or as a scalar (the magnitude of $\vec{a}$). There are many applications of vectors in science and engineering. How does a top spin? How was Pluto discovered? How did Astronaut White maneuver in space? How is a rocket guided? Vectors are used in the solution of each of these problems.

In mathematics an algebra of vectors has been developed. You have started to build an understanding of vector algebra. You have added and subtracted vectors. You have divided a vector by a scalar. Several other properties are considered in the exercises.

4–4 Exercises Acceleration Vectors

1. What are three ways a driver may accelerate the car he is driving?
2. A rocket accelerates uniformly at 100 ft/sec^2 for 10 seconds starting from rest. What is its velocity at the end of 5 seconds and at the end of 10 seconds? How far did the rocket travel in 5 seconds and 10 seconds? Compare the velocities and distances at the end of 5 seconds and at the end of 10 seconds. Do the comparisons appear reasonable?

4–5 Acceleration of Falling Objects

It was Sir Isaac Newton who first proposed the requirements for putting a satellite in orbit about the Earth. He came to his conclusions by analyzing motion and arriving at a descriptive language for motion such as the one we have used. His conclusions about orbital motion were an extension of an analysis of the motion of projectiles.

You should remember that a predecessor of Newton's, Galileo, studied the motion of falling bodies and shocked the population of his day by stating that all bodies fall with similar motions. Today we know that this motion is best represented by an acceleration vector.

Figure 4–12 shows a vector representation of the velocity of a falling object at one second intervals. Notice that for $t > 0$ the differences $(\vec{v}_{t+1} - \vec{v}_t)$ between any two successive vectors are the same. The time interval between successive vectors is one second. Then the acceleration is

$$\vec{a} = \frac{\vec{v}_{t+1} - \vec{v}_t}{\Delta t} = \frac{\overrightarrow{\Delta v}}{\Delta t}$$

Since $\Delta t = 1$ and for a particular location on Earth $\overrightarrow{\Delta v}$ is the same for each value of t, the vector $\vec{a}$ does not change and is designated $\vec{g}$.

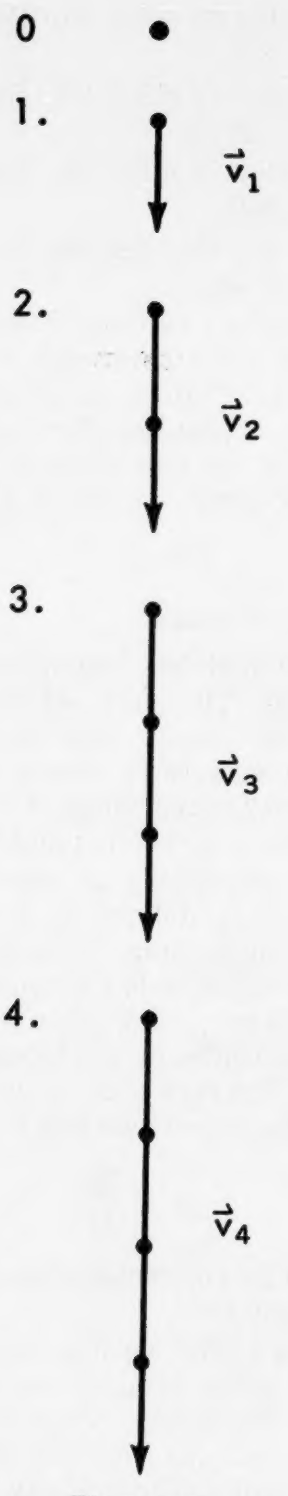


Figure 4–12

Before continuing with projectile motion we shall re-examine the frame of reference idea to see if there is a way of showing that $\vec{g}$ is always "down."

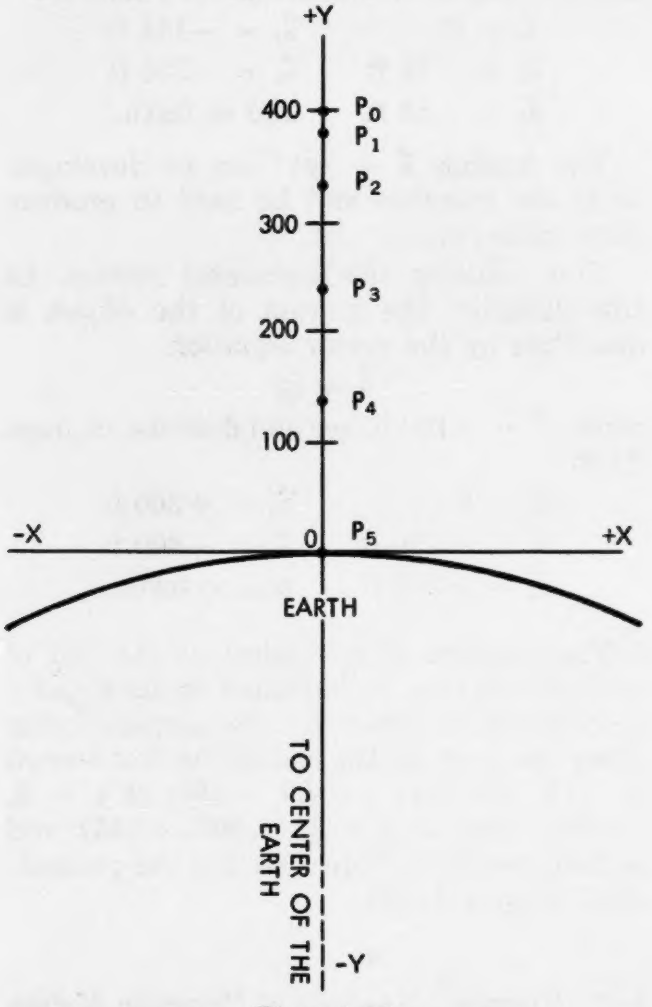


Figure 4-13

At the location of the object at 0 seconds (before it begins to fall) the lines indicating the frame of reference may be drawn as shown in Figure 4-13. The x-axis indicates the horizon; the y-axis indicates the vertical. The points P_0, P_1, P_2, P_3, P_4, P_5 indicate the positions of the object at 1 second intervals of fall. Notice that P_t is the position after t seconds. The object starts falling from P_0 and reaches the Earth 5 seconds later.

The positions of the points may be indicated by their y-coordinates; they may also be considered in terms of displacements from the origin of the frame of reference. Then we use the sign to indicate direction and have in feet:

$$\overrightarrow{OP}_0 = +400 \qquad \overrightarrow{OP}_3 = +256$$
$$\overrightarrow{OP}_1 = +384 \qquad \overrightarrow{OP}_4 = +144$$
$$\overrightarrow{OP}_2 = +336 \qquad \overrightarrow{OP}_5 = +0$$

We may now find the change in displacement $\overrightarrow{\Delta s}_t$, the average velocity $\bar{\vec{v}}_t$, and the acceleration $\vec{a}_t$ for each one second interval. For example, during the third second

$$\begin{aligned}\overrightarrow{\Delta s}_3 &= \overrightarrow{OP}_3 - \overrightarrow{OP}_2\\ &= (+256) - (+336) = -80 \text{ ft.}\\ \bar{\vec{v}}_3 &= \frac{\overrightarrow{\Delta s}_3}{\Delta t} = \frac{-80}{1} = -80 \text{ ft/sec.}\end{aligned}$$

During the fourth second

$$\begin{aligned}\overrightarrow{\Delta s}_4 &= \overrightarrow{OP}_4 - \overrightarrow{OP}_3\\ &= (+144) - (+256) = -112 \text{ ft.}\\ \bar{\vec{v}}_4 &= \frac{\overrightarrow{\Delta s}_4}{\Delta t} = \frac{-112}{1} = -112 \text{ ft/sec.}\\ \vec{a}_4 &= \frac{\bar{\vec{v}}_4 - \bar{\vec{v}}_3}{\Delta t}\\ &= \frac{(-112) - (-80)}{1} = -32 \text{ ft/sec}^2.\end{aligned}$$

The value -32 ft/sec² is the common approximation for $\vec{g}$ at locations near sea level. In Chapter 5 you will read about the factors which affect $\vec{g}$ at any location in space.

4-6 Analysis of Projectile Motion

Now let us return to a look at projectile motion. We shall use both velocity vectors and acceleration vectors. Suppose an object is on a high cliff and by some means is put in motion horizontally so that it leaves the cliff with a horizontal velocity of 100 ft/sec. As soon as the object leaves the support of the cliff and is in space, gravity will cause it to accelerate downward at 32 ft/sec².

The origin of the frame of reference is placed at the top of the cliff where the object starts its motion (Figure 4-14). For $t > 0$ the velocity vector $\vec{v}_t$ is in the $+x$ direction and the acceleration vector $\vec{a}$ is in the $-y$ direction. Newton's contribution to the solution of this type of problem was to suggest that the motions in the two directions could be handled separately. Each may be used to establish one element of the ordered pair necessary to locate the object within the frame of reference at any time t.

Since the time of motion is limited by the length of time it takes the object to reach the ground, we shall examine motion in the $-y$ direction first. How far is the object displaced in the $-y$ direction at the end of each second. We can use the result established in previous sections and compute the

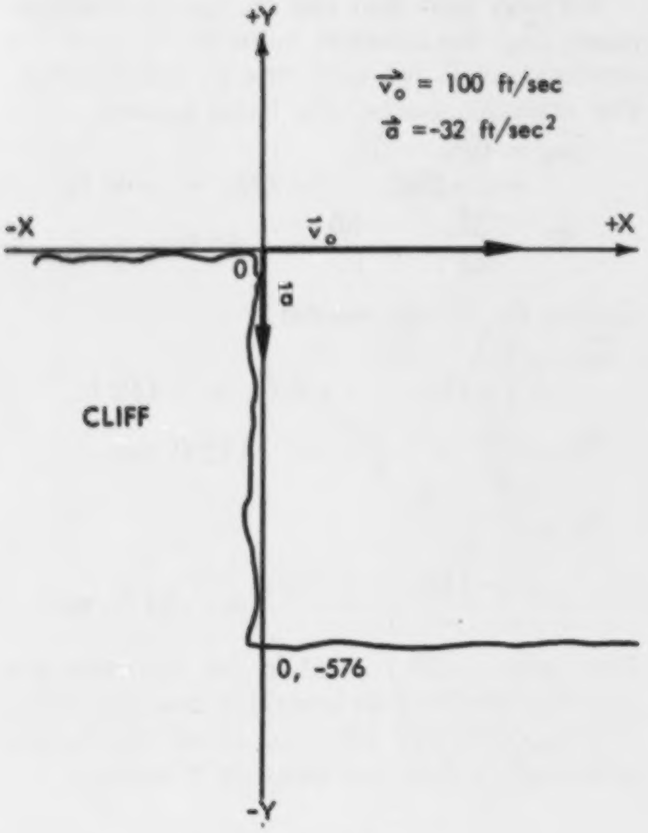


Figure 4-14

displacement from a knowledge of acceleration and time. Remember that for one second intervals

$$\vec{\Delta s}_t = \vec{s}_{t+1} - \vec{s}_t$$
$$\Delta t = 1$$
$$\vec{v}_t = \frac{\vec{s}_{t+1} - \vec{s}_t}{\Delta t} = \frac{\vec{\Delta s}_t}{\Delta t}$$
$$\vec{a}_t = \frac{\vec{v}_{t+1} - \vec{v}_t}{.\ \Delta t} = \frac{\vec{\Delta v}_t}{\Delta t}$$

When we consider the −y direction, $\vec{v}_0 = 0$ since the only initial velocity is in the +x direction; we say that the y component of the initial velocity is 0. We also have $\vec{s}_0 = 0$, $t_0 = 0$ and $\vec{a} = -32$ ft/sec² for all values of t. The velocity in the y direction at the end of the first second is −32 ft/sec; the average velocity during the first second is $\frac{0 + (-32)}{2}$; that is, −16 ft/sec. Since $\Delta t = 1$ we have for $t > 1$

$$-32 = \vec{v}_{t+1} - \vec{v}_t \qquad (t > 1)$$
$$\vec{v}_{t+1} = \vec{v}_t - 32$$

and the values of $\vec{v}_t$ are

$\vec{v}_0 = 0$ $\vec{v}_3 = -80$ ft/sec
$\vec{v}_1 = -16$ ft/sec $\vec{v}_4 = -112$ ft/sec
$\vec{v}_2 = -48$ ft/sec and so forth.

Similarly since for $\Delta t = 1$

$$\vec{v}_t = \vec{s}_{t+1} - \vec{s}_t$$
$$\vec{s}_{t+1} = \vec{s}_t + \vec{v}_t$$

and the displacements along the y-axis are

$\vec{s}_0 = 0$ $\vec{s}_3 = -144$ ft
$\vec{s}_1 = -16$ ft $\vec{s}_4 = -256$ ft
$\vec{s}_2 = -64$ ft and so forth.

The formula $\vec{s} = \frac{1}{2}\vec{a}t^2$ can be developed as in the exercises and be used to produce these same results.

Now consider the horizontal motion. In this direction the motion of the object is described by the vector equation

$$\vec{s} = \vec{v}t$$

where $\vec{v} = +100$ ft/sec and does not change. Then

$\vec{s}_0 = 0$ $\vec{s}_3 = +300$ ft
$\vec{s}_1 = +100$ ft $\vec{s}_4 = +400$ ft
$\vec{s}_2 = +200$ ft and so forth.

The position of the object at the end of each second may be indicated by its x and y coordinates as given by the displacements along the axes. At the end of the first second (t = 1) we have (+100, −16); at t = 2, (+200, −64); at t = 3, (+300, −144); and so forth until the object strikes the ground. (See Figure 4-15).

4-6 Exercise Analysis of Projectile Motion

Refer to Figure 4-15 and consider the trajectory (the path) of the object.

Interpolation is a process whereby the scientist guesses what is going on in his problem between the points he has plotted. Before you use interpolation to guess the trajectory of the object, extend the data by finding the coordinates for the half second intervals. Then connect the points to picture the path of the object.

What two factors determined the horizontal distance traveled?

Suppose the height (altitude) were increased to several thousand feet, what changes would have to be made in our unstated assumption about ground level? (Magellan's trip settled this, didn't it?) Go way up to satellite altitude (hundreds to thousands of miles) and guess what might happen.

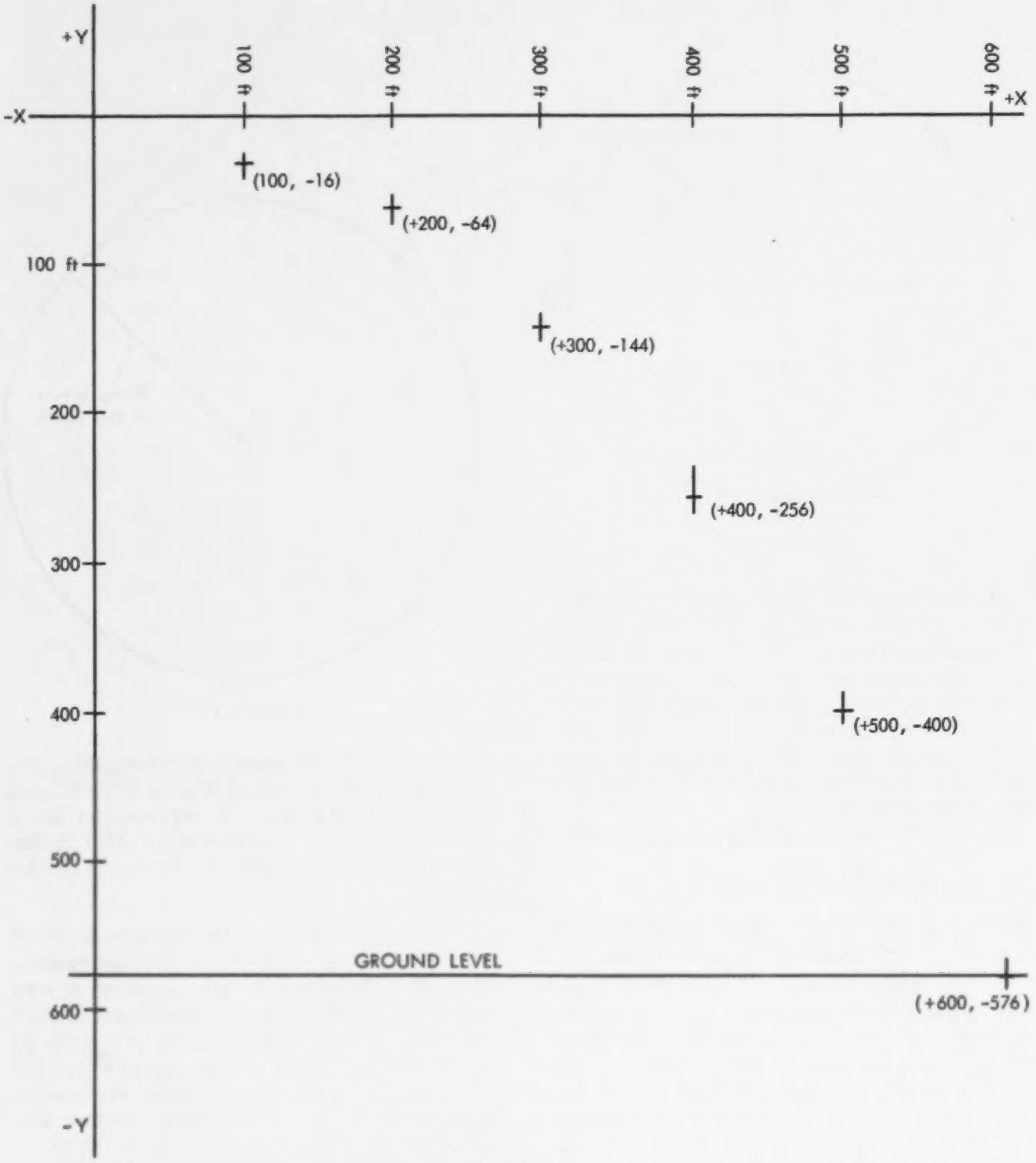


Figure 4–15

4–7 Circular Motion

We have just used velocity and acceleration vectors to describe the motion of an object on a curved path (Section 4–6). Let us now examine the motion of an object around a circular path and seek a way of describing this motion in terms of vector quantities.

Any point P_1 on a circle may be identified by its coordinates (x_1, y_1) or by the radius, r, and $\angle XOP_1$ where the angle is measured counter-clockwise as in Section 1–3. In Figure 4–16 $\angle XOP_1 = 45°$. For any angle θ and any point P: (x, y) on the circle

$$x = r \sin \theta,$$
$$y = r \cos \theta.$$

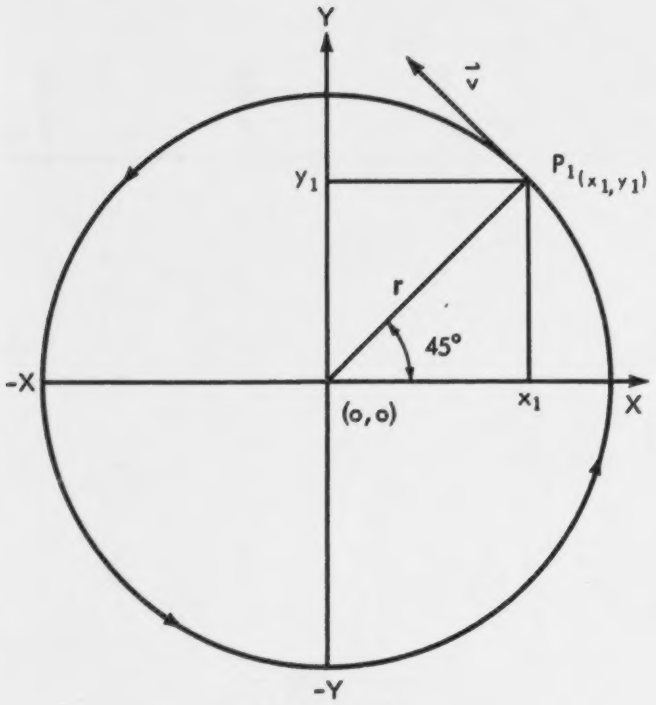


Figure 4-16

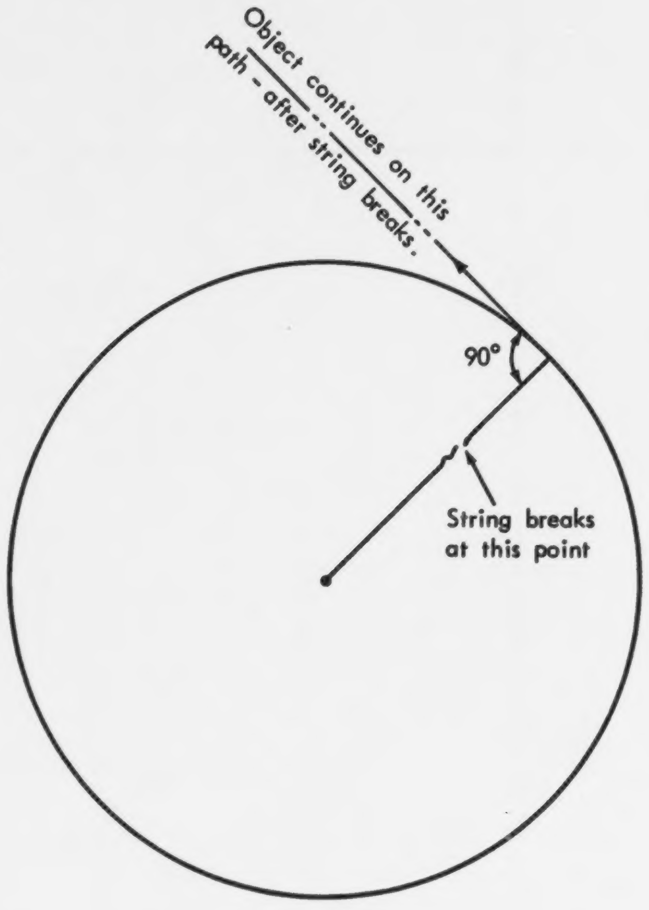

Object continues on this path - after string breaks.
90°
String breaks at this point

Figure 4-17

A circular motion of an object is repetitive; that is, after one revolution the same motion is repeated. This fact can be used to determine the speed of the object. (Is speed a vector or a scalar quantity?) Suppose r = 3 ft; then the circumference, c, of the circle is $2\pi r$ where $\pi \approx 3.14$; $c \approx 18.8$ ft. Suppose also that the object takes 5 seconds for 10 revolutions. Then

$$d = 188 \text{ ft. (for 10 revolutions)}$$

$$v = \frac{d}{\Delta t} \approx \frac{188}{5} \approx 37.6 \frac{\text{ft}}{\text{sec}}$$

where v is used for the (scalar) magnitude of $\vec{v}$ to avoid confusion of rate and radius.

If the object were moving at 37.6 ft/sec in a straight line and the direction was known we could represent its velocity by a vector. To do this with circular motion would seem to violate the concept of vectors for the object's direction of motion is always changing. Is it ever going in a +y, or −x, or some other direction? The answer may seem to be no. We know, from laboratory evidence, that there is a tie or force toward the center of the circle which holds the object on its circular path. If this tie were severed the object would immediately assume a straight line path which has a direction tangent to the circle at the point where the object was located when the tie was broken (Figure 4-17). For this reason and others, which are developed in advanced mathematics, the velocity of the object in Figure 4-16 at any point P on the circle is represented by a vector having a magnitude of 37.6 ft/sec and a direction tangent to the circle at the point P.

Figure 4-18 is a vector representation of the velocities of an object in circular motion. The velocity vectors are equally spaced about the circle. A vector equivalent to each of these velocity vectors is drawn with its origin at the center of the circle; $\vec{v}_1 = \vec{v}_1'$, $\vec{v}_2 = \vec{v}_2'$, $\vec{v}_3 = \vec{v}_3'$, and so forth. We use the vectors $\vec{v}_1'$, $\vec{v}_2'$, . . . to obtain vectors $\overrightarrow{\Delta v_1}$, $\overrightarrow{\Delta v_2}$, and so forth.

Notice the geometric figure formed by the $\overrightarrow{\Delta v}$ vectors. Suppose that the equal spacings between the vectors on the circumference were decreased until there were many, many velocity vectors on the circumference and each had an equivalent drawn from the center of the circle. Then the $\overrightarrow{\Delta v}$ vectors would (1) increase in number and (2) decrease in magnitude. Now what would the geometric picture of the $\overrightarrow{\Delta v}$ vectors begin

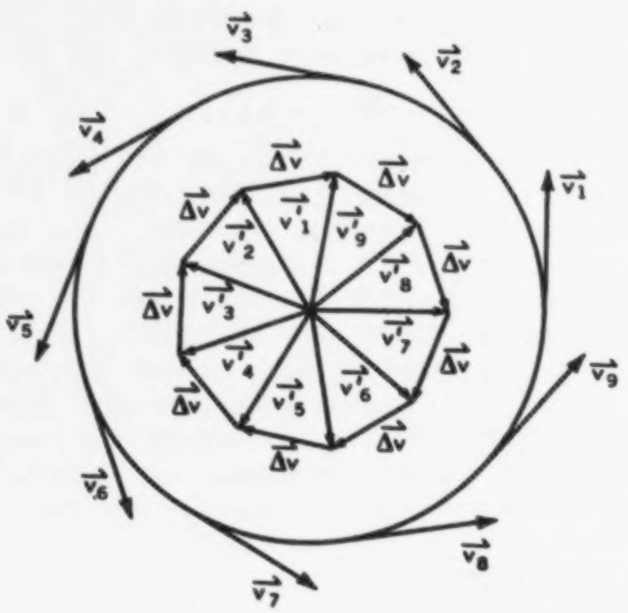


Figure 4–18

to look like? The figure would approach a circle in appearance. This is a key idea. If it doesn't make sense to you, then you should make a vector drawing following the directions given until you see the circle. Remember that the speed, v, is the same at all points of the circle.

The sum of the $\overrightarrow{\Delta v}$'s will approximate a circle with a radius of v. Therefore the change in velocity in one revolution of the object has a magnitude of $2\pi v$, the circumference of the circle made by the $\overrightarrow{\Delta v}$'s.

Consider our example in which 10 revolutions took 5 seconds. One revolution occurs in 0.5 seconds and the acceleration of our object will be

$$a = \frac{2\pi v}{t} = \frac{2\pi 37.6}{0.5} \approx 574 \text{ ft/sec}^2$$

The time for one revolution of an object in a circular motion is referred to as the period T of the motion. Thus for circular motion we have

$$v = \frac{2\pi r}{T} \text{ and}$$

$$a = \frac{2\pi v}{T} = \frac{2\pi(2\pi r/T)}{T} = \frac{4\pi^2 r}{T^2}$$

4–7 Exercise Circular Motion

From $a = \frac{4\pi^2 r}{T^2}$ and $v = \frac{2\pi r}{T}$ prove that $a = \frac{v^2}{r}$.

4–8 Angular Velocity

There is another way to look at the speed of an object in circular motion. The object considered in Section 4–7 was timed at 0.5 seconds for one revolution. This statement could be restated as 360° per 0.5 seconds or changed to unit time 720°/sec. In physical and mechanical problems it is common to state angles in terms of radians (Section 1–3) and to call the speed in terms of angles per unit time *angular velocity.* It is customary to use the symbol ω (Greek letter omega) for the angular velocity in

$$\frac{\text{radians}}{\text{unit time}}$$

Then since 1 revolution is 2π radians we have

$$\omega = \frac{2\pi}{0.5} = 4\pi\frac{\text{rad}}{\text{sec}}$$

Notice that this corresponds to 720 deg/sec since 2π radians equal 360°.

The conversion from angular velocity ω to linear velocity v is possible since

$$s = r\theta$$

and therefore

$$\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$

whence

$$v = r\omega$$

The angular velocity $\omega = 4$ rad/sec then corresponds to a linear velocity

$$v = 4\pi r \approx 12 \times 3.14 \approx 37.6 \text{ ft/sec}$$

as we obtained in Section 4–7.

4–8 Exercise Angular Velocity

What is the angular velocity of your location on Earth? How does this compare with the angular velocity of the location of a person living on the equator? For this problem assume that Earth rotates once every 24 hours and has a radius of 4000 miles. You may have to look on a map to find your latitude. How does your speed compare with the speed of a person on the equator? You may use trigometric functions or proportions to get the radius and circumference of the circle of latitude for your location.

Do you see how this type of consideration effected the placing of the Atlantic Missile Range in Florida rather than Maine?

4–9 Forces

We have considered ways of describing motion throughout the first eight sections of this chapter. In other words we have considered kinematics, the mathematical and graphical description of motion. The kinematics used by scientists in describing the motions of satellites and space vehicles is more complex than these descriptions. It requires the competent use of plane and solid geometry, analytical geometry, trigonometry, vector algebra and calculus. Many mathematicians are employed at NASA locations, such as the Goddard Space Flight Center, to determine the kinematics of space craft.

How is the motion of a space vehicle produced and controlled? What effects motion? Even though the story has been told many times, it is worthwhile (particularly in this space age) to ponder over the behavior of moving objects as described by Newton. (Often referred to as Newton's laws of motion.)

Everyday experiences seem to indicate that all moving objects eventually will stop moving. Questions arise since the time to stop may vary. What causes objects to stop? Why does a ball roll longer on a bare floor than on a rug? Why does a tennis ball slow down faster than a baseball? You could make up many similar questions. It seems that something called a force (friction, drag) is present in every "slowing down" process.

The motion of an object at any instant can be described by a velocity vector. Due to the inertia of the object, it will continue with this same velocity vector unless some force is applied to it.

If a force is applied to an object then its motion will change. If the force is in the same direction as the object is moving, the object will gain speed; if the motion and force are in opposite directions the object will slow down. This change can also be represented by vectors. The longer the force is applied the greater the change in velocity. Indeed the product of the force and the time of application is proportional to the change in velocity produced; we write $\vec{F}\Delta t \propto \overrightarrow{\Delta v}$. After units have been specified, we shall have

$$\vec{F}\Delta t = k\overrightarrow{\Delta v}$$

for some constant k. Notice the force $\vec{F}$ must be a vector quantity since $\overrightarrow{\Delta v}$ is a vector, both k and Δt are scalars, and the product of a scalar and a vector is a vector.

This mathematical reasoning can be verified by laboratory experimentation. If you carry the thought a little further, you will realize that the directional part of $\vec{F}$ must be preserved in the $\overrightarrow{\Delta v}$ because the scalar quantity, Δt, can only change the magnitude of $\vec{F}$. Therefore the direction of the change in velocity must be in the direction of the applied force.

Notice that a knowledge of the mathematics of vectors is a very valuable tool in arriving at a hypothesis about the behavior of physical objects. Then the hypothesis can be checked (hopefully confirmed) in the laboratory. Thus a tremendous amount of trial and error experimentation can be eliminated.

4–10 Mass

Suppose you mistook a bowling ball for a soccer ball and gave it a swift kick. Neglect the pain you would experience and examine the motion produced on the ball. Would the bowling ball behave in the manner you expected from the soccer ball? Why not?

We observed in Section 4–9 that since objects have inertia, an outside force must be applied to an object to change its velocity. If equivalent forces are applied for the same time interval to several totally different objects the resulting changes in velocity may not be the same. The quantity of inertia each object possesses is called the inertial mass (often just called mass) of the object. To a physicist or space scientist, the mass of an object is a measure of the object's resistance to a change of velocity by a force.

An experiment will demonstrate this idea of mass. For this experiment you will need:

a sturdy table or work bench,
a vise,
hack saw blades,
"C" clamps of several different sizes,
a stop watch or clock with a second hand.

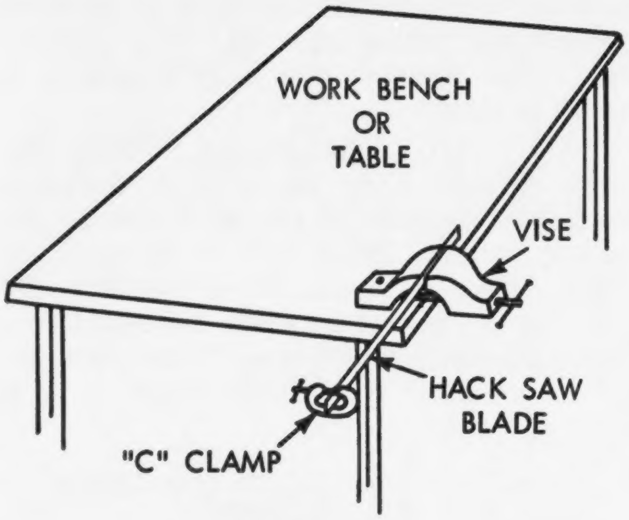


Figure 4–19

Clamp one end of a hack saw blade in the vise. On the other end fasten the smallest "C" clamp. Cause the "C" clamp to swing back and forth by initially deflecting it a few inches and letting go. Time the interval for 10 complete swings of the "C" clamp. (Any number of swings will do as long as you can adequately time that number.) Repeat this for a "C" clamp of each size. Note your results in a Table with headings as in Table 4–3.

Table 4–3

"C" clamp size	**Total number swings**	**Time for total swings**	**Frequency Number of swings per unit time**	**Ratio of $\frac{n}{\Delta t}$ of each to first clamp**	**Ratio of masses**
smallest					
largest					

The *frequency*, number of swings per unit time, is an indication of the response of the inertial mass of each "C" clamp to the force supplied by the spring of the hack saw as it stops and starts the clamp at the end of each half swing. A comparison of the masses of the clamps can be made by comparing the frequencies. Use the smallest clamp as the basis for comparison and find the ratio of the masses of the clamps to it. The swinging hack saw blade and clamp is an example of an inertial pendulum.

Numerous experiments of many types have led to the conjecture that the rate of change of velocity (that is, acceleration) is related to the mass. If the mass is doubled, then the acceleration is halved; if the mass is multiplied by three, then the acceleration is one third as much as before. We say that acceleration varies inversely as mass; that is,

$$a \,\alpha\, \frac{1}{M}$$

Since from Section 4–9

$$\vec{F}\Delta t = k\overrightarrow{\Delta v}$$

$$\vec{F} = k\frac{\overrightarrow{\Delta v}}{\Delta t} = k\vec{a}.$$

The statement $a \,\alpha\, 1/M$ implies that the product of the mass and the magnitude of the acceleration is a constant; that is, $Ma = k$. This constant may be combined with the constant k in the equation $\vec{F} = k\vec{a}$ to give us

$$\vec{F} = k'M\vec{a}$$

for some constant k′. For an appropriate choice of units of measure

$$\vec{F} = M\vec{a}$$

which is the customary form for Newton's second law of motion.

4-10 Exercise Mass

Equivalent impulses ($\vec{F} x \Delta t$) are applied to 2 space vehicles. One has twice the change of velocity of the other. What does this difference in behavior convey about their masses?

4-11 Units of Measure

In order to avoid chaos in the meaning of units of measure, international organizations have formulated and accepted definitions of units. In this country the National Bureau of Standards cooperates with agencies of other countries and establishes the units that we use. Some confusion exists because there are two systems in use throughout the world. The *English System* is used in daily living in the United States. Most of the rest of the world and scientists everywhere use the *metric system*. Table 4-4 shows the basic units in both systems. The purpose here is not to define precisely or to compare these units, but to establish how each one is used. Notice that the unit for mass in the English system and the unit for force in the metric system are printed in capital letters to emphasize that each of these must be understood in terms of the other three units in its particular system.

Table 4-4

Basic Units of Measure

	Length	Mass	Time	Force
English	foot	SLUG	second	pound
Metric	meter	kilogram	second	NEWTON

In the English system you are acquainted with the foot, second, and pound as units. If we use these units in the formula $F = Ma$, consider only the magnitudes of the quantities (Section 4-10), and solve for M we have

$$M = \frac{F}{a}$$

where the magnitude F of $\vec{F}$ is measured in pounds and the magnitude a of $\vec{a}$ is measured in ft/sec². Thus the unit designation of mass in the English system could be called

$$\frac{\text{lb}}{\text{ft/sec}^2}$$

In order to reduce the wordiness of "pounds per foot per second squared," it is called a slug. Thus mass in the English system is stated in slugs.

Your *weight* is the amount of force the mass of your body exerts in a direction towards the center of the earth due to the pull of gravity. What is your mass? What is the mass of a person whose weight is 128 lb? Near sea level the person is subject to a force of gravity which would accelerate the person at 32 ft/sec². Since weight w is a force we have:

$$M = \frac{F}{a} = \frac{w}{g} = \frac{128\text{ lb}}{32\text{ ft/sec}^2} = 4\text{ slugs}$$

Notice that $w = Mg$. Therefore the weight of a given mass changes when there is a change in the value of g. If the force of gravity could be zero, the weight would be zero.

Astronaut White was in a condition of "apparent" weightlessness during his "walk in space." This phenomenon is explained on pages 132-33. He maneuvered himself by gently tugging on the tether cord connecting him to the space craft. Suppose that the combined weight of White and his equipment, if he was on Earth would be 192 pounds. Then the mass that he needs to move by tugging on the tether cord is 192/32; that is, 6 slugs. If he had no initial velocity toward the spacecraft, what *impulse* ($F \times \Delta t$) would be needed to propel White back to the Gemini IV at 1/2 ft/sec? Let's consider our formulas:

$$F \times \Delta t = m\Delta v = 6 \times \tfrac{1}{2} = 3$$

The magnitude of the impulse (that is, the product of the applied force and the time interval during which it is applied) must be 3. A 3 pound force applied for one second is equivalent to a 6 pound force applied for one-half second.

It is interesting that Astronaut McDivitt mentioned that he found it necessary to make compensating corrections in the space craft's position because of White's tugs. Does it make sense that when a tug is made on one end of a cord which is fastened at the other end that the fastened end exerts the same magnitude of force on the object to which it is tied? The Gemini space craft weighs (on Earth) 7000 pounds. Can you find the velocity of the space craft caused by the tug?

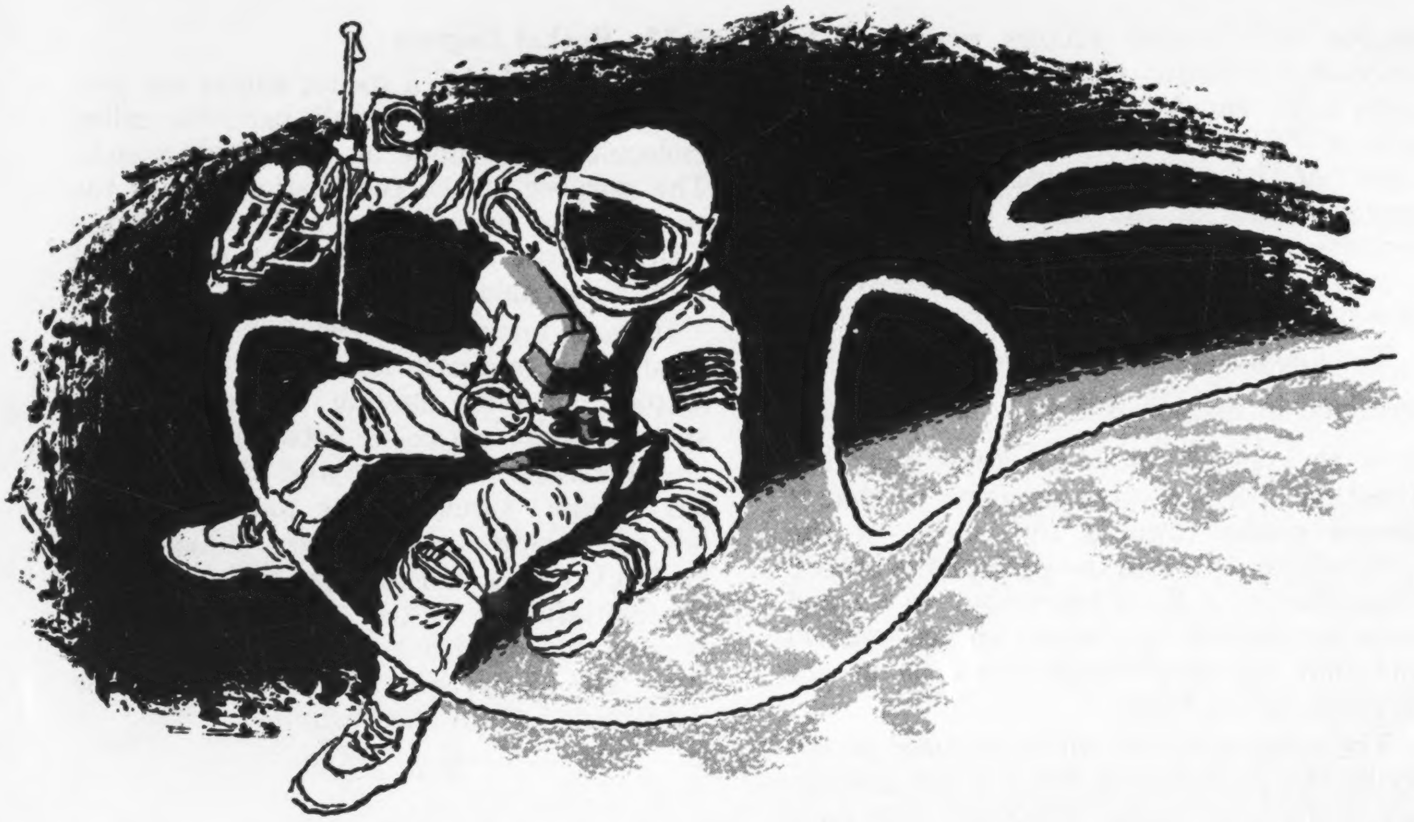


Figure 4-20

This problem has been presented as if the Gemini space craft had zero velocity. Actually it is moving at very high speeds. We have used the space craft as the center of our frame of reference and the frame of reference moved with the craft. Hence when White and the space craft had the same velocity, the relative velocity of the two was zero. This technique is common in the solution of many problems.

Let us now refer to Table 4-4 again and look at the metric units. If you are not familiar with the meter, and kilogram, you should ask any science teacher to let you see a meter stick and a kilogram mass; that is, a unit of mass. To get the units for force we again examine the formula $F = Ma$. When M is expressed as kilograms and a as meters/second squared then F must be $\frac{\text{kilogram meters}}{\text{sec}^2}$. To reduce the wordiness of "kilogram meters per second squared," it is called a *newton*, (abbreviated nt.).

In the metric system weight should be expressed in newtons since weight is a force. The metric value of g is 9.8 m/sec^2. A kilogram of material at the surface of Earth weighs 9.8 newtons.

On several occasions we have stated that all objects at one location fall with the same acceleration. This should seem reasonable after it is noted that the weight w of an object is dependent on the mass of the object. We have $W = Mg$ from the formula $F = Ma$. Therefore, W/M is a constant at any one location; that is, weight is directly proportional to mass. Any change in mass will be accompanied by a proportional change in weight.

4-11 Exercises Units of Measure

1. In the laboratory we determine the mass of an object by weighing it on a balance. This means we balance the weight of an unknown mass with the weight of a known mass. Why does this tell us the mass of the unknown?
2. A force of 10 newtons is applied for 5 seconds to a 10 kilogram mass. What is the resulting change in velocity? What is the acceleration during the 5 second interval?

4-12 Dynamics of Circular Motion

Circular motion is a special case of an object traveling in a curvaliniar path. In

Section 4–7 velocity vectors were used to describe the motion of an object at a given point in its circular path. The velocity vectors at different points had different directions but the same magnitude. For equally spaced points of the circle the successive changes of velocities in equal time intervals provided acceleration vectors that were equal in magnitude.

The magnitude of the acceleration was determined in Section 4–7 as $a = \frac{2\pi v}{T}$ where T is the period. Now we know that the direction of $\vec{a}$ is always changing so that it always points towards the center of the circle which describes the path of the object. From Section 4–9 we know that an applied force is required to produce an acceleration and that the acceleration must be in the direction of the force.

The characteristics which we used to describe the acceleration vector must also apply to the force vector. The force must have constant magnitude and always be directed toward a fixed point, the center of the circular path. Any force that has these two characteristics is called a *central force.*

Any circular motion may be described as motion that can be represented by a velocity having uniform magnitude and a central force. The magnitude of the central force can be computed from the mathematical statement of Newton's Second Law of Motion and magnitude of the acceleration in circular motion:

$$F = Ma, \qquad a = \frac{v^2}{r}, \qquad F = \frac{Mv^2}{r}$$

The paths of satellites, both natural and man-made, usually are not perfectly circular. For a circular orbit the velocity at a point must be directed perpendicular to the force at the point. Also the magnitude of the velocity and the force must satisfy a precise relationship. The force is provided by the Earth gravitational field in the case of Earth-orbiting satellites and thus depends in part upon altitude. So space scientists must be able to construct, launch, and control a satellite and its launching vehicle so that it will have a precise altitude, speed, and direction at the instant the space craft is to assume a circular orbit. If any one of these requirements (parameters) is not attained, the orbit will not be truly circular.

4–13 Rocket Engines

The forces inside a rocket engine are produced by extremely small particles called molecules that move at very high speeds. The purpose of the combustion within the engine is to provide the high temperature which causes the molecules to move in a very, very rapid fashion.

In your imagination you can picture the result of a fantastic number of sub-microscopic particles moving at tremendous speeds. Collisions occur between molecules and between molecules and the walls of the engine. Consider the molecules of a hot gas inside a closed tank. When a gas molecule comes to rest upon colliding with the container wall the change in velocity is equal in magnitude to the original velocity. The force of the collision can be determined from the formula

$$\vec{F} = \frac{M\overrightarrow{\Delta v}}{\Delta t}$$

where M is the mass of the molecule, Δt is the time of the collision and Δv is the change of velocity. (The product of MΔv is called *momentum.*)

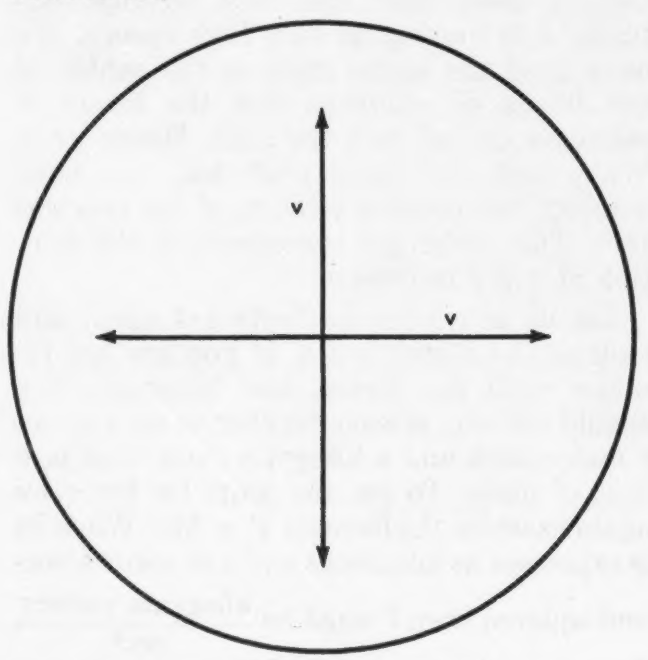


Figure 4–21

Figure 4–21 is a representation of the forces acting on the inside of a container where for simplicity the third dimension has been omitted. All forces are of the same magnitude. Then the sum of the force vectors must be zero. (You should be able to visualize this.) The container has no motion because the forces are in *equilibrium.* (Figure 4–22)

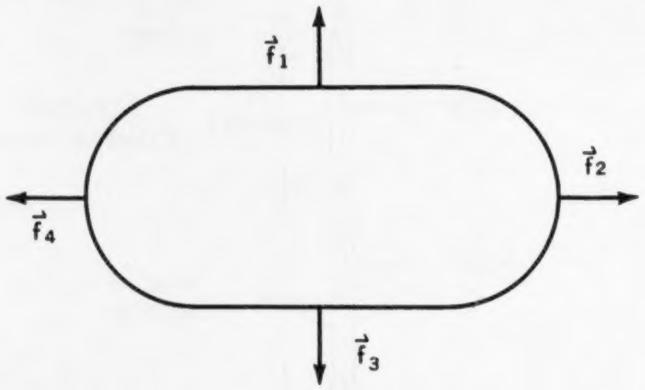


Figure 4–22

Let us remove a section from one end of the container and make an opening in the end as pictured in Figure 4–23. The vector picture is now changed because the molecules moving in the direction of the open end do not collide with the container. Instead these molecules of gases escape from the open end. There can be no force exerted on the container in the direction of the open end by the internal gases. The gases rushing out the open end represent only those molecules having a velocity in that particular direction. The other force vectors are not affected. Now the sum of the forces as pictured in Figure 4–23 are not in equilibrium. There is a net force $\vec{F}_2$ upon the end of the container opposite the opening and thus the container will move in that direction.

The magnitude of the force $\vec{F}_2$ which moves the container can be found by determining the momentum of the molecules of gas which escape out of the open end. It is these molecules which are *not* balancing the propulsion force. In order to find the force $\vec{F}_2$ we need to know (1) the velocity of the escaping gas molecules, (2) the mass of each molecule, (3) the number of molecules, and (4) the time interval.

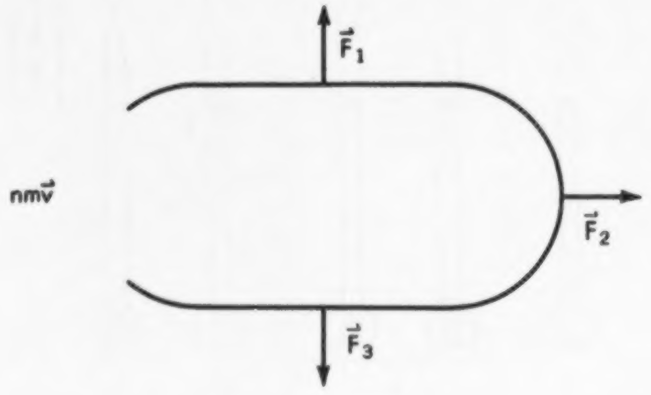


Figure 4–23

The velocity of the escaping gas molecules can be measured by appropriate instruments. The total mass of the escaping molecules is simply the total mass of the burning mixture supplied to the container. The time interval is the time in which the total mass is burned. From these data the force can be computed using the same formula as before:

$$\vec{F} = \frac{M\vec{v}}{\Delta t}$$

In rocketry this is the force which is called *thrust.* The names and thrusts for a few of the launch vehicles used by NASA are listed in Table 4–5.

Table 4–5

Launch vehicle Name	Stage	Thrust pounds
Thor-Agena	1 2	170,000 16,000
Atlas-Agena	1 2	368,000 16,000
Titan II	1 2	430,000 100,000
*Saturn V	1 2 3	7,500,000 1,000,000 200,000

* *Scheduled for first launch in 1967.*

4–13 Exercise Rocket Engines

A 50,000 pound thrust acting on a rocket for 5 seconds produces a 4,000 ft./sec change of velocity. What will be the change of velocity produced by a 10,000 pound thrust acting on the rocket for 5 seconds? Would it be possible for a 10,000 pound thrust to produce a 4,000 ft./sec change of velocity on the same rocket?

4–14 Sounding Rockets

Among the many space exploration programs that NASA administers are those which investigate that realm of space below which Earth orbiting satellites travel and above which balloon ascension is not possible.

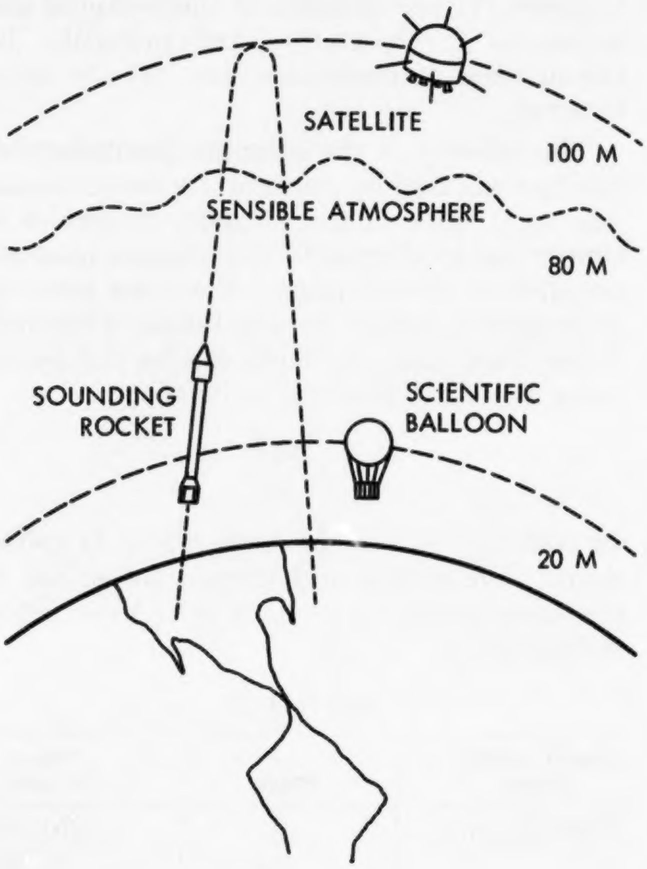

SATELLITE
100 M
SENSIBLE ATMOSPHERE
80 M
SOUNDING ROCKET
SCIENTIFIC BALLOON
20 M

Figure 4–24

This involves a type of vehicle known as a sounding rocket.

The Nike Apache is a sounding rocket. The name indicates that the payload of scientific instrument is launched by a two rocket motor system (Figure 4–25). The Nike booster burns for 3.5 seconds with a thrust of 42,500 pounds. The Apache motor provides a 5,130 thrust for 6.4 seconds.

During a launch optical and radar instruments track the movement of the rocket. The tracking data is handled by computer that not only records the numerical value of such factors as time intervals, velocity, altitude, range, and angles of the flight paths but also prepares the graphic displays of these data as shown on Figure 4–26 through 30 at the end of this section (pages 114-118).

Many forces effect the flight of the rocket. The two obvious ones are gravity and the thrust of the rocket motors. In addition the atmosphere effects the flight by the movement of winds and air resistance; that is, *drag.* Figure 4-31 shows several configurations of sounding rockets which have differing degrees of drag. Notice that the Figures 4–26 through 30 are labeled "Drag Case I."

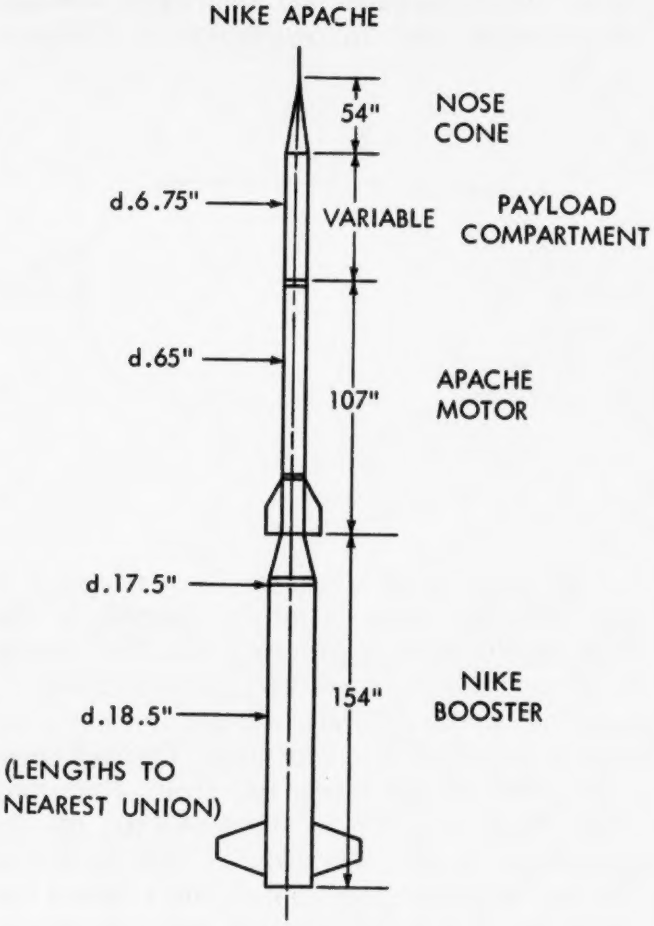

NIKE APACHE
54"
NOSE CONE
d.6.75"
VARIABLE
PAYLOAD COMPARTMENT
d.65"
107"
APACHE MOTOR
d.17.5"
154"
NIKE BOOSTER
d.18.5"
(LENGTHS TO NEAREST UNION)

Figure 4–25

The rocket under consideration was launched at Wallops Island, Virginia, at an angle of 80° to the horizontal in order that the point of impact would be in the waters of the Atlantic ocean and not over inhabited land.

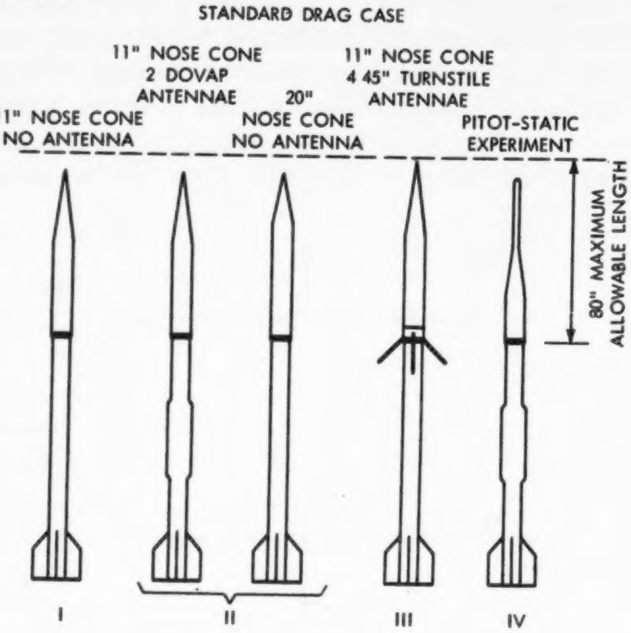

STANDARD DRAG CASE
11" NOSE CONE NO ANTENNA
11" NOSE CONE 2 DOVAP ANTENNAE
20" NOSE CONE NO ANTENNA
11" NOSE CONE 4 45" TURNSTILE ANTENNAE
PITOT-STATIC EXPERIMENT
80" MAXIMUM ALLOWABLE LENGTH
I
II
III
IV

Figure 4-31

Table 4–6

Phase number	Name of phase	Time of flight
1	Nike Burning	0 to 3.5 sec
2	Coasting	3.5 to 20.0 sec
3	Apache Burning	20.0 to 26.4 sec
4	Burn-out to Apogee	26.4 to 198.5 sec
5	Apogee to Impact	198.5 to 385.8 sec

The flight of the rocket is divided into 5 parts as shown in Table 4–6. The term burning means that the rocket motor is providing a thrust in the manner described in Section 4–13.

To know the composite picture of the rockets' behavior the interrelation of the graphs (Figure 4–26 through 30) must be understood. Figures 4–26, 4–27, 4–28, and 4–29 have the same time base (x coordinate) for Altitude, Horizontal Range, Flight Path Angle, and Velocity. Figure 4–30 shows a combination of the altitude and horizontal range data but the scales for the x-axis and y-axis are not the same. Accordingly, the graphs must not be read as a picture. Replotting Figure 4–30 with equal x and y scales would give a more realistic idea of the geometry of the trajectory of the rocket.

The following questions and exercises are to assist you in gaining scientific information from the graphs.

4–14 Exercises Sounding Rockets

1. What is the maximum altitude (apogee) of the rocket? What is the average vertical velocity up to apogee?
2. How far did the rocket travel horizontally (range) by the time it reached apogee? What was the rockets' average horizontal velocity during this interval?
3. From your answers for Exercise 1 and 2 find the true average velocity (speed and direction) for the interval.
4. How far from the launch site was the point of impact? (total range)
5. What is the average horizontal velocity of the rocket?
6. From Figures 4–26 and 4–27 determine the rockets' true velocity at 90 seconds. Verify your answer (magnitude and direction) from the data on Figure 4–28 and 4–29. (Suggestion: Use a 20 second interval to determine ΔS).
7. Identify each of the five phases with patterns of velocity in Figure 4–29.
8. During the second phase (coasting) Figure 4–29 shows a decrease in velocity. Determine the value of the negative acceleration for this interval. How does your answer compare with the value of g?
9. What average acceleration was produced by each of the two rocket engines?
10. In Figure 4–28, the curve of the flight path angle goes through 0 and changes from a positive to a negative sign. How do you interpret the change in sign?
11. In order to know the speed and direction of the rocket at any point in flight, which two graphs must be used?

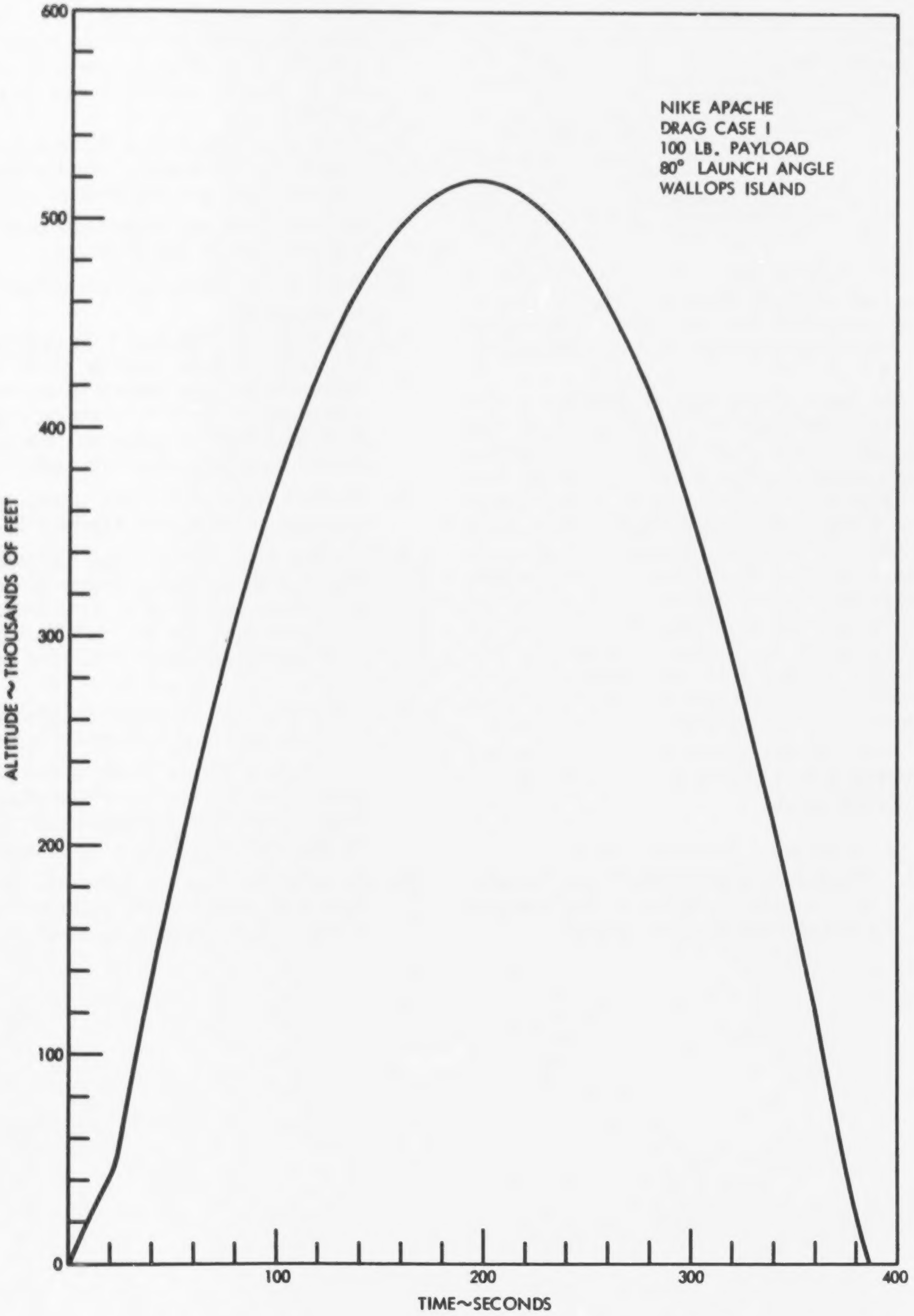

NIKE APACHE
DRAG CASE I
100 LB. PAYLOAD
80° LAUNCH ANGLE
WALLOPS ISLAND
ALTITUDE ~ THOUSANDS OF FEET
600
500
400
300
200
100
0
100
200
300
400
TIME~SECONDS

Figure 4-26

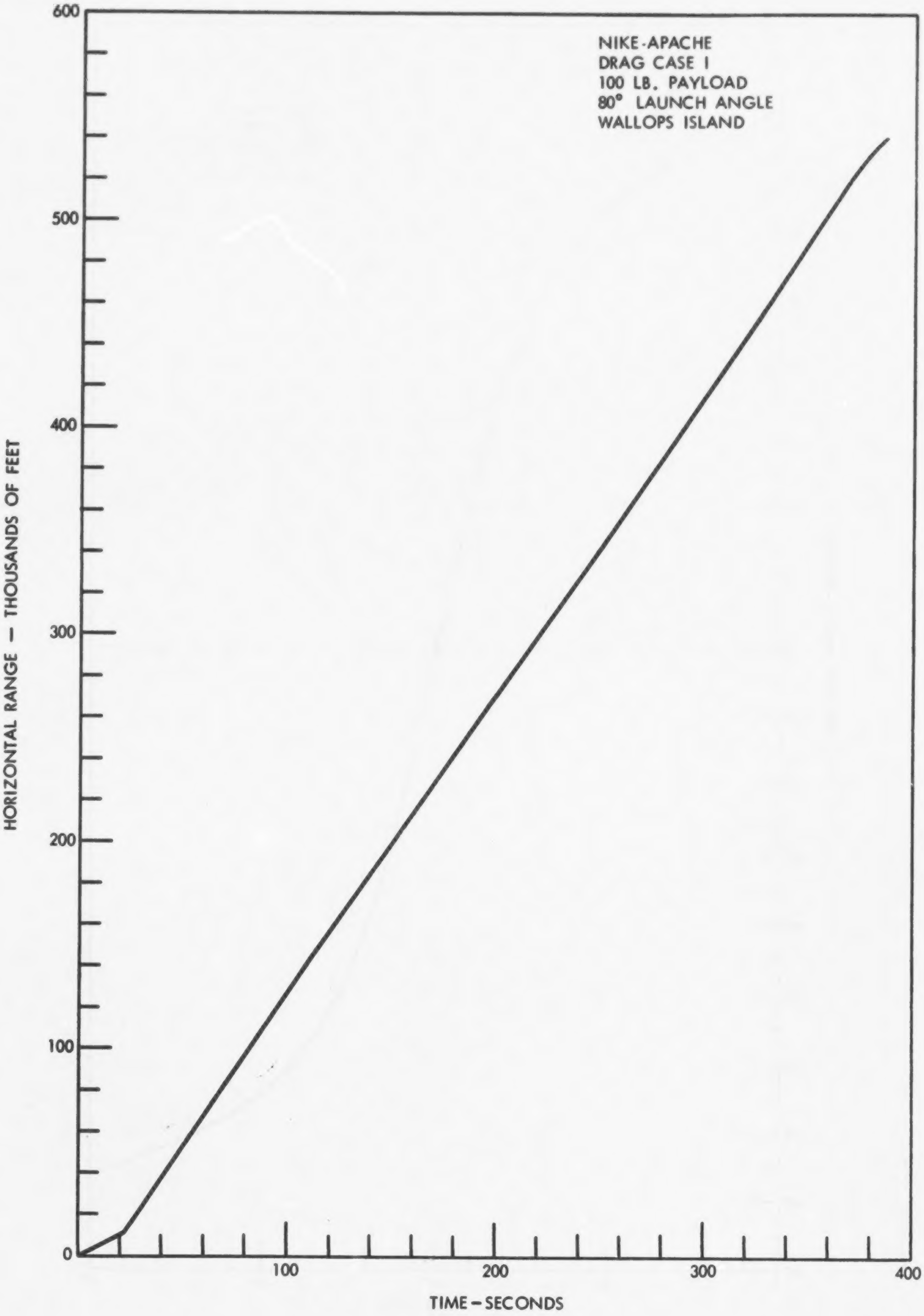

NIKE-APACHE
DRAG CASE I
100 LB. PAYLOAD
80° LAUNCH ANGLE
WALLOPS ISLAND
HORIZONTAL RANGE — THOUSANDS OF FEET
600
500
400
300
200
100
0
TIME – SECONDS
100
200
300
400

Figure 4-27

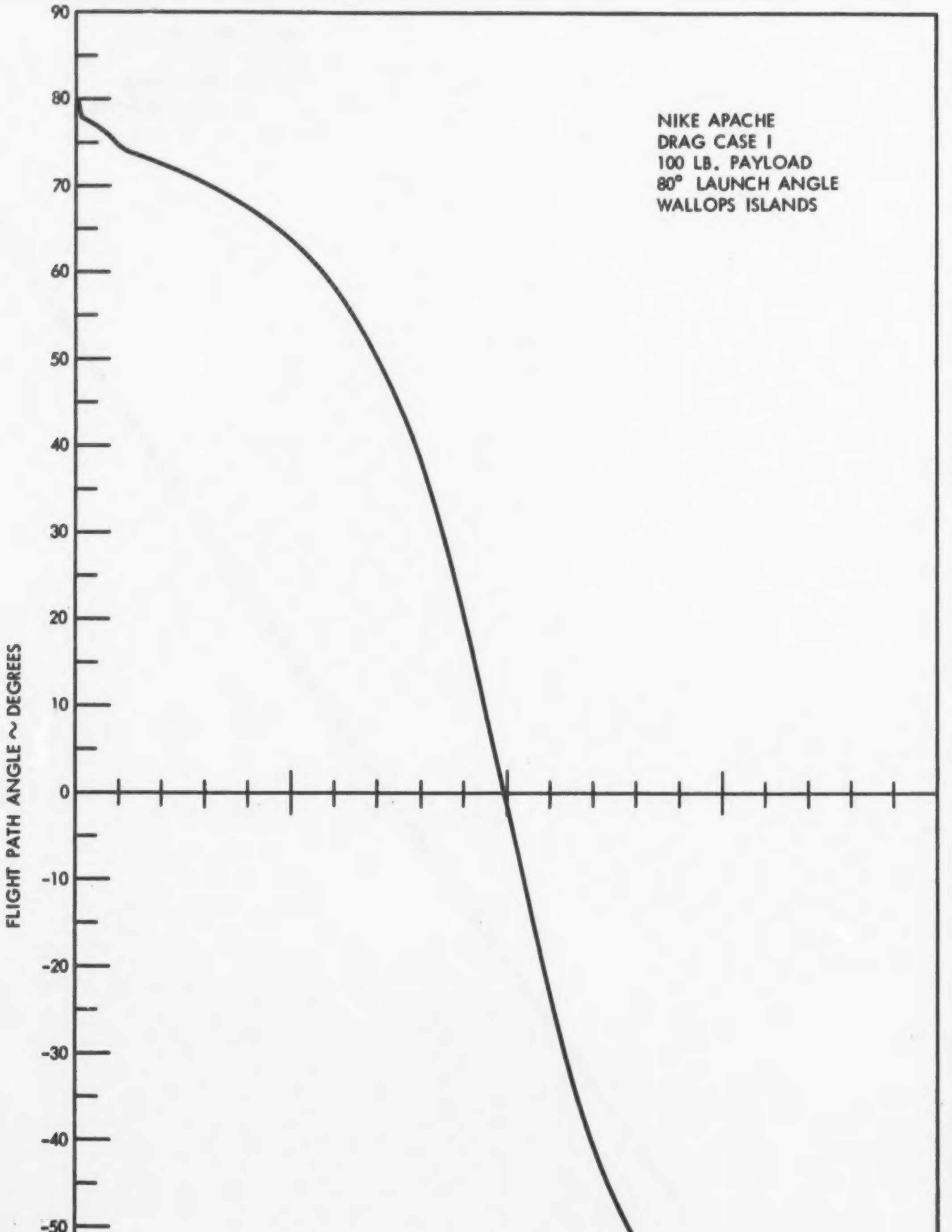
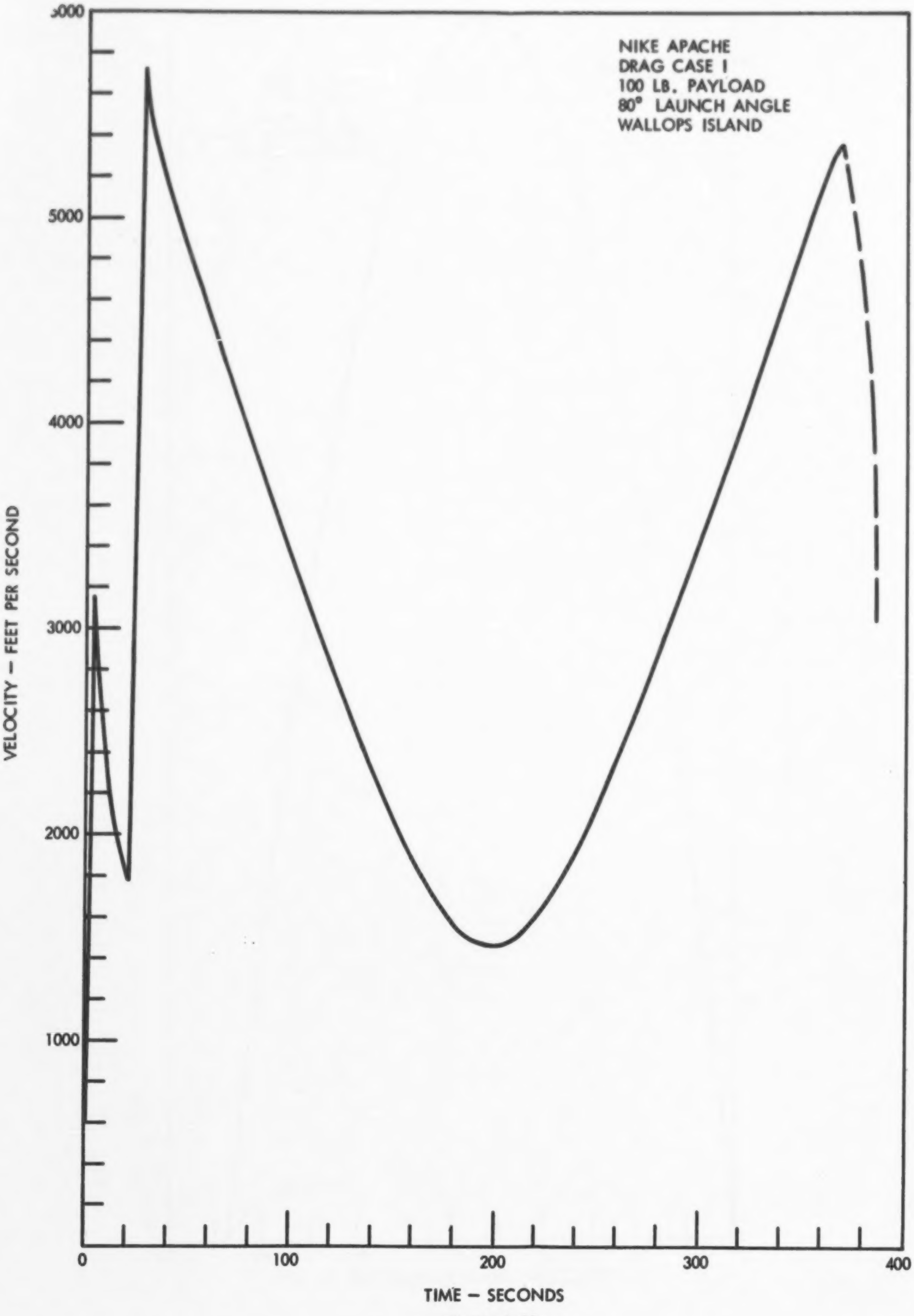

NIKE APACHE
DRAG CASE I
100 LB. PAYLOAD
80° LAUNCH ANGLE
WALLOPS ISLANDS
FLIGHT PATH ANGLE ~ DEGREES
90
80
70
60
50
40
30
20
10
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
0
100
200
300
400
TIME ~ SECONDS

Figure 4-28



NIKE APACHE
DRAG CASE I
100 LB. PAYLOAD
80° LAUNCH ANGLE
WALLOPS ISLAND
VELOCITY – FEET PER SECOND
6000
5000
4000
3000
2000
1000
0
100
200
300
400
TIME – SECONDS

Figure 4-29

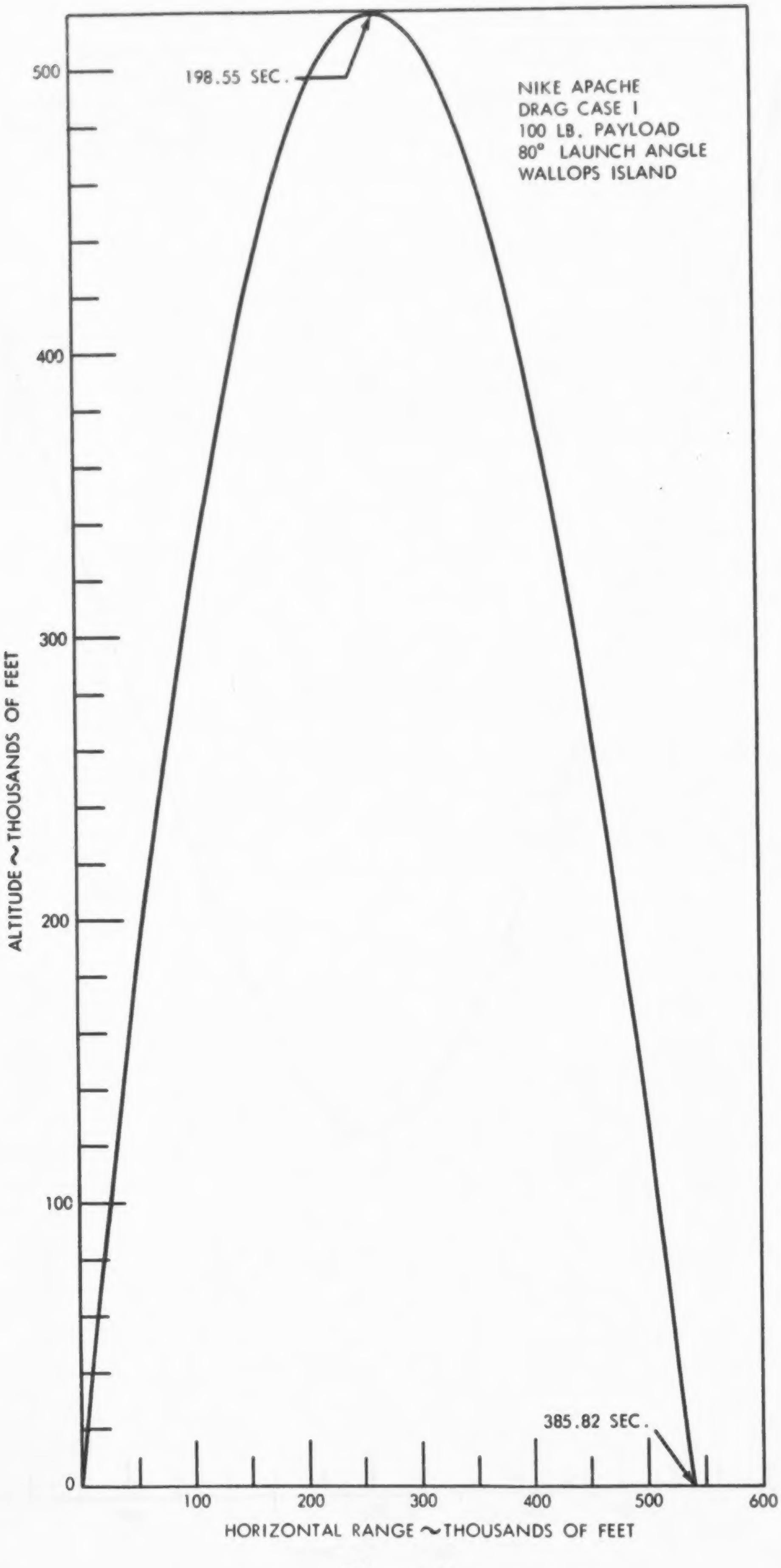
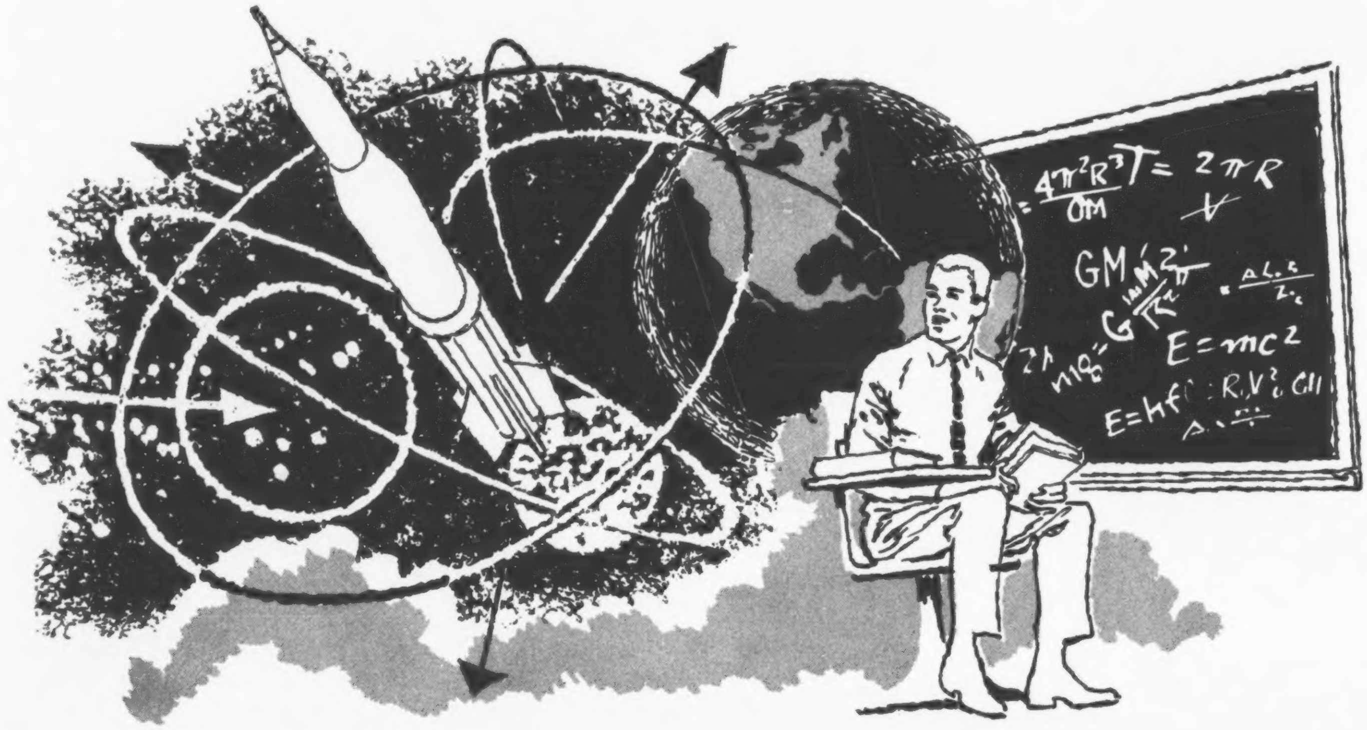

198.55 SEC.
NIKE APACHE
DRAG CASE I
100 LB. PAYLOAD
80° LAUNCH ANGLE
WALLOPS ISLAND
385.82 SEC.
ALTITUDE ~ THOUSANDS OF FEET
0
100
200
300
400
500
100
200
300
400
500
600
HORIZONTAL RANGE ~ THOUSANDS OF FEET

Figure 4-30


$= 2\pi R$
GM
$E=mc^2$

Chapter 5

SPACE MECHANICS

by
Robert F. Chambers
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Newark, Delaware

The remaining chapters of this publication are especially designed to furnish supplementary activities in space-science oriented mathematics for students who have had an opportunity to participate in the advanced courses now being offered in many of our high schools. However, all students will find these materials informative and interesting in that they provide an insight into the new course content of high school curricula.

SPACE MECHANICS

SPACE MECHANICS has been written for students who are especially interested in mathematics and science. Its purpose is to answer some of the many questions about space flight and demonstrate the application of mathematics to space problems. Two mathematical concepts that may be new to you have been introduced to serve as examples of new and exciting mathematical concepts which lie ahead. SPACE MECHANICS has been prepared to help bridge the gap between mathematics as it is taught formally in the classroom and as it is used by physicists and engineers. When you work the exercises, be sure to keep your units of measure straight or you may find yourself in orbit around Hogan's barn.

5–1 Velocity and Acceleration

Figure 5–1 illustrates the setup a student used to record the displacement of a car as it was pulled along a horizontal wire by a steady force. Mass m (about 610 grams) and the car (about 1000 grams) are coupled with an appropriate length of paper tape. Cash register tape about $1\frac{1}{8}$ inches wide is suitable for this purpose. The tape should pass freely through the timer and over the double pulley. The portion of the tape that is passing through the timer is covered with a strip of carbon paper. This is held in place with wicket shaped wires as illustrated in Figure 5–2. The carboned surface should face down. Soon after the timer is started the car should be released. The falling mass will pull the car along the wire. The vibrating clapper will strike the moving tape and the impact marks will show how the position of the car changed with time. The rubber stopper is a shock absorber and makes it possible to stop the car without placing undue strain on the wire. The stopper should have room to slide after impact. The turnbuckle is used to draw the wire taut.

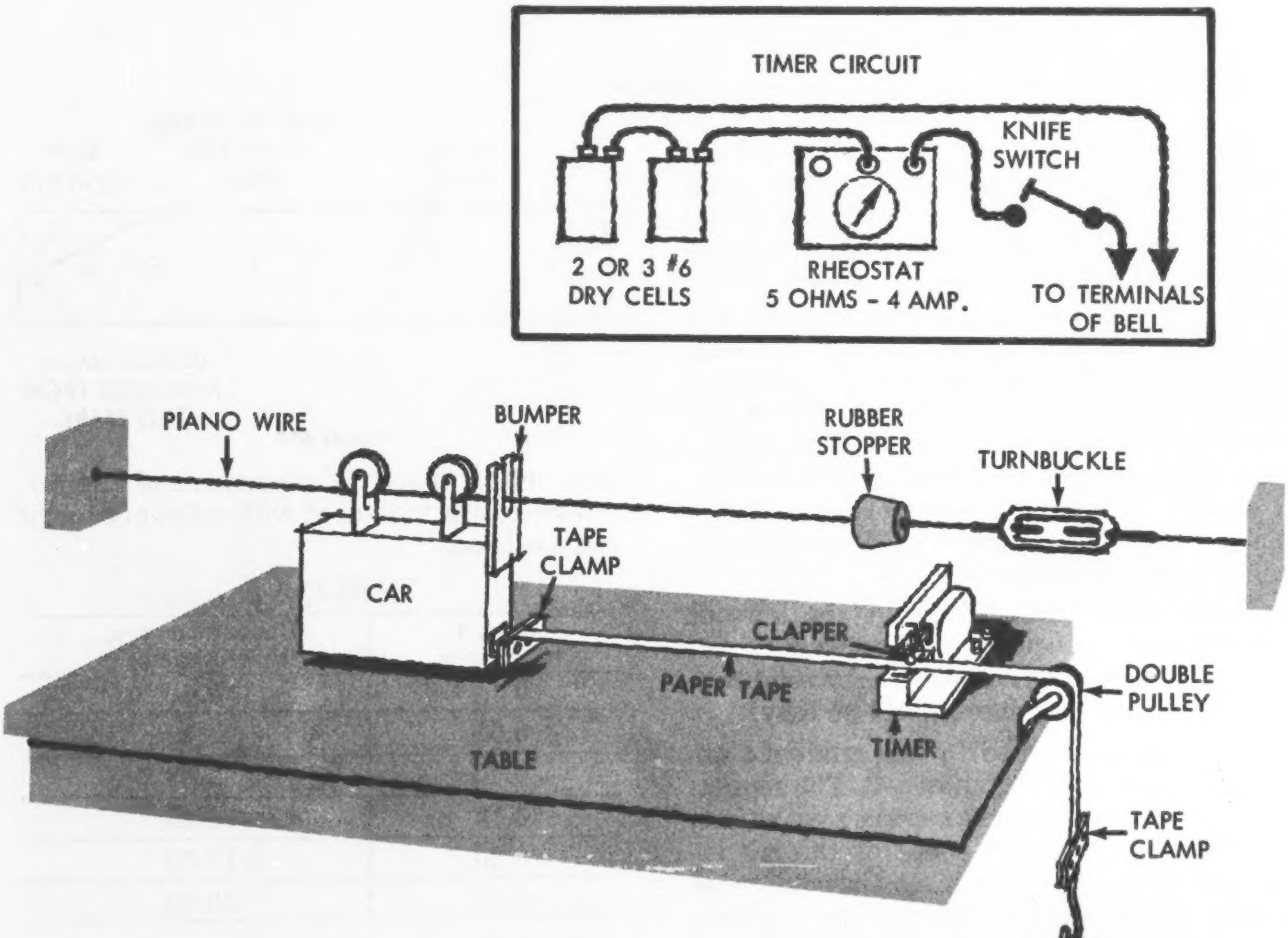

TIMER CIRCUIT
KNIFE SWITCH
2 OR 3 #6 DRY CELLS
RHEOSTAT 5 OHMS - 4 AMP.
TO TERMINALS OF BELL
PIANO WIRE
BUMPER
RUBBER STOPPER
TURNBUCKLE
TAPE CLAMP
CAR
CLAPPER
PAPER TAPE
TIMER
DOUBLE PULLEY
TABLE
TAPE CLAMP
m
MASS

Figure 5–1

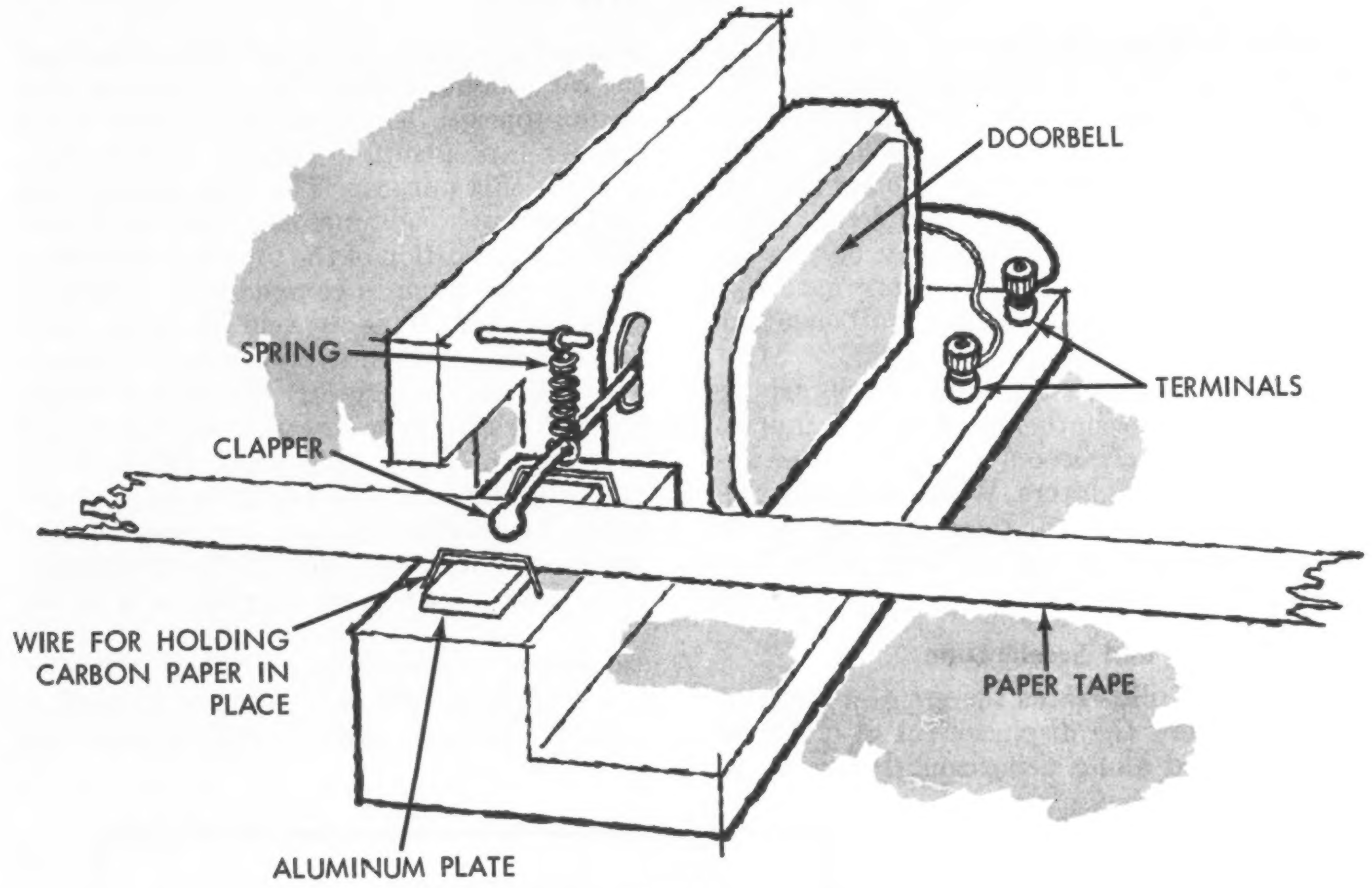

DOORBELL
SPRING
TERMINALS
CLAPPER
WIRE FOR HOLDING CARBON PAPER IN PLACE
PAPER TAPE
ALUMINUM PLATE

Figure 5-2

The timer (Figure 5-2) is a modified doorbell. To be of any value as a timer the motion of its clapper must be essentially periodic; that is, the time intervals for all successive strikes of the clapper must be essentially the same. The setting of the contacts and the operating voltage are critical for periodic operation of a doorbell clapper. These factors are also peculiar to the doorbell. Hence the conditions needed to achieve periodic operation of the clapper are found experimentally. The time between successive strikes of the clapper can be measured with an instrument called a stroboscope.

Much can be learned about the laws of motion with the equipment described. It can be built at a modest cost. You may need help with the timer but the rest will be easy.

The tape as marked by the student's experiment is shown in Figure 5-3. The separation of the impact marks increases with time, an indication that the speed of the car increases with time. Realizing that he could not be sure about the time for the displacement from rest to the first impact mark, the student measured time with reference to the first impact mark. Displacement of the car, however, was measured with reference to the rest position.

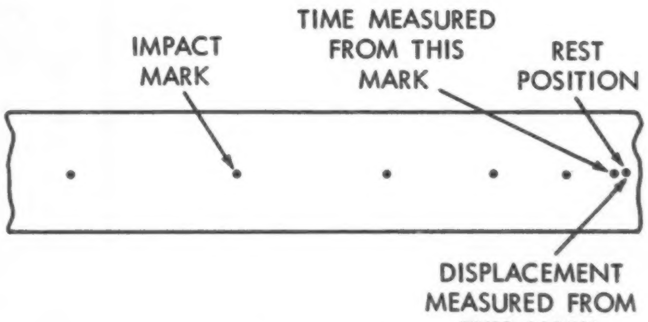

IMPACT MARK
TIME MEASURED FROM THIS MARK
REST POSITION
DISPLACEMENT MEASURED FROM THIS MARK

Figure 5-3

TABLE 5-1

Time, t in seconds	Displacement, s in centimeters
0	0.48
0.05	2.40
0.10	5.85
0.15	10.80
0.20	17.30
0.25	25.20

Table 5-1 contains the data collected by the student.

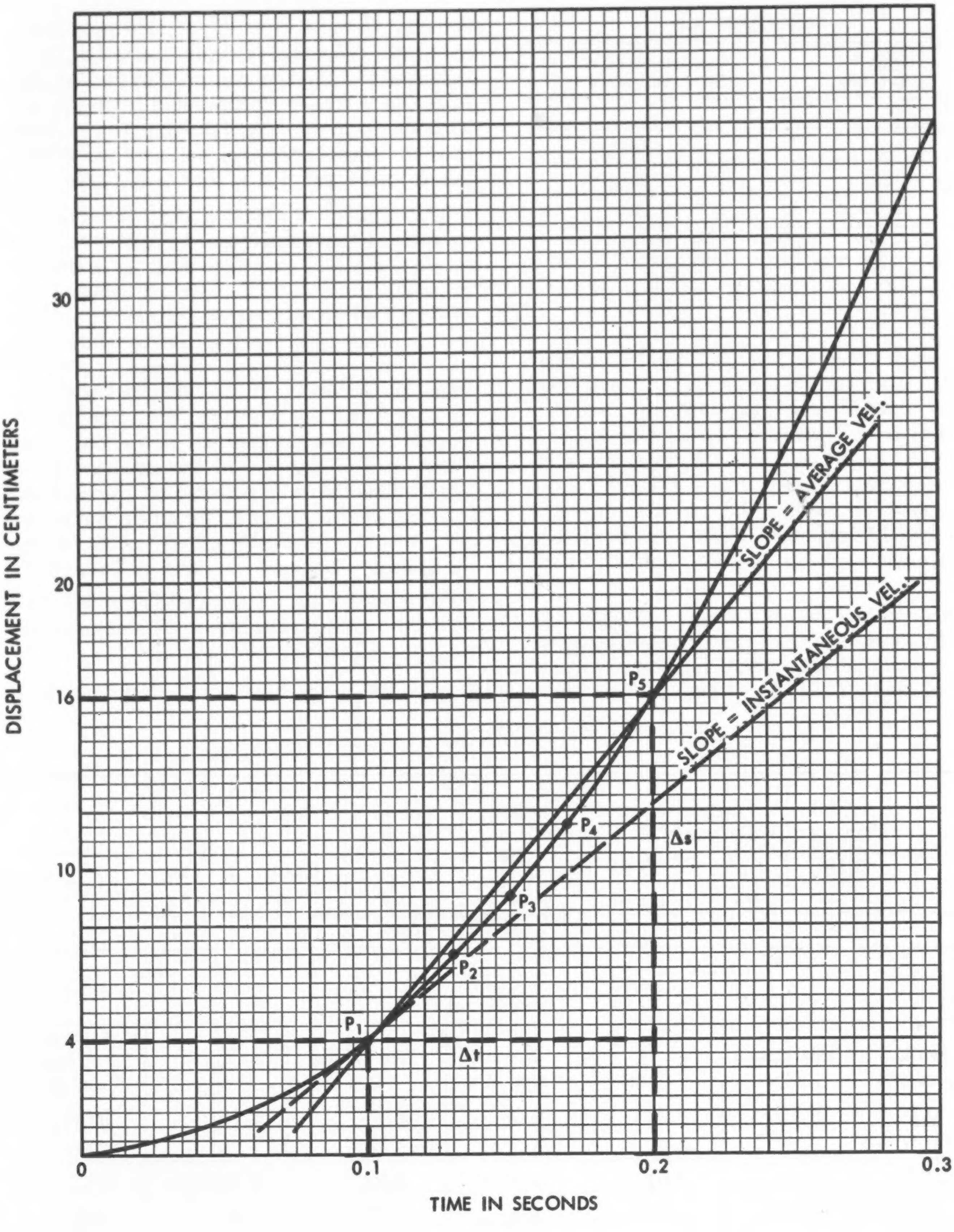

DISPLACEMENT IN CENTIMETERS
30
20
16
10
4
0
0.1
0.2
0.3
TIME IN SECONDS
P_1
P_2
P_3
P_4
P_5
Δt
Δs
SLOPE = AVERAGE VEL.
SLOPE = INSTANTANEOUS VEL.

Figure 5–4

TABLE 5-2

Time, t in seconds	Displacement, s in centimeters
0	0
0.10	4.00
0.20	16.00
0.30	36.00
0.40	64.00
0.50	125.00
0.60	144.00

Table 5–2 is ideal data for a motion similar to that studied by the student. The period of the timer for the ideal data is 0.1 seconds whereas it was 0.05 seconds for the student. The ideal data has been introduced for convenience only. Figure 5–4 is a graph of the ideal data for the first 0.3 seconds. This will be referred to as the student's graph since the experimental graph has the same general form.

As a part of the laboratory exercise, the student was required to determine the average velocity of the car during several intervals of time. This he did using the average velocity formula $\vec{v} = \overrightarrow{\Delta s}/\Delta t$. He read from his graph (Figure 5–4) values of s for given values of t. This data was used to find Δs and Δt so the average velocity could be computed with the formula. The data that the student used to compute the assigned average velocities from Figure 5–4 is found in Table 5–3.

The student recognized that the average velocity of the car decreased markedly as the interval of time was decreased by reducing the final value of t. This trend aroused his curiosity so he wanted to pursue it further. He could see that there would be some difficulty in reading the graph accurately for real small values of Δt, so he consulted with his teacher about this problem. The student and teacher working together discovered that Figure 5–4 is a graph of the equation $s = 400\ t^2$. This information enabled the student to calculate the value of s for any given value of t. Armed with the equation for his curve, the student obtained the data in Table 5–4.

After analyzing the data of Table 5–4, the student came to three important conclusions.

These were:

1. As the new value of t decreases and

TABLE 5-3

Initial Value of t	Final Value of t	Δt	Initial Value of s	Final Value of s	Δs	Average Velocity Δs/Δt
0.10	0.20	0.10	4.00	16.00	12.00	120
0.10	0.17	0.07	4.00	11.56	7.56	108
0.10	0.15	0.05	4.00	9.00	5.00	100

TABLE 5-4

Initial Value of t	New Value of t	Δt	Initial Value of s	New Value of s	Δs	Average Velocity Δs/Δt
0.10	0.13	0.03	4.00	6.76	2.76	92.0
0.10	0.12	0.02	4.00	5.76	1.76	88.0
0.10	0.11	0.01	4.00	4.84	0.84	84.0
0.10	0.105	0.005	4.00	4.41	0.41	82.0
0.10	0.103	0.003	4.00	4.2436	0.2436	81.2
0.10	0.102	0.002	4.00	4.1616	0.1616	80.8
0.10	0.101	0.001	4.00	4.0804	0.0804	80.4
0.10	0.1005	0.0005	4.00	4.0401	0.0401	80.2
0.10	0.1001	0.0001	4.00	4.008004	0.008004	80.04

approaches 0.10, the values of Δt, Δs, and the average velocity decreases.

2. As t approaches 0.10 the increment Δt is becoming smaller and approaching zero as a limit. The value of Δt can be made as near zero as you want but it cannot be zero for $t \neq 0.10$.
3. As Δt approaches zero as a limit, the magnitude of the average velocity ($\Delta s/\Delta t$) becomes smaller and appears to approach 80 as a limit. The average velocity can be made as near 80 as you want but it cannot be 80.

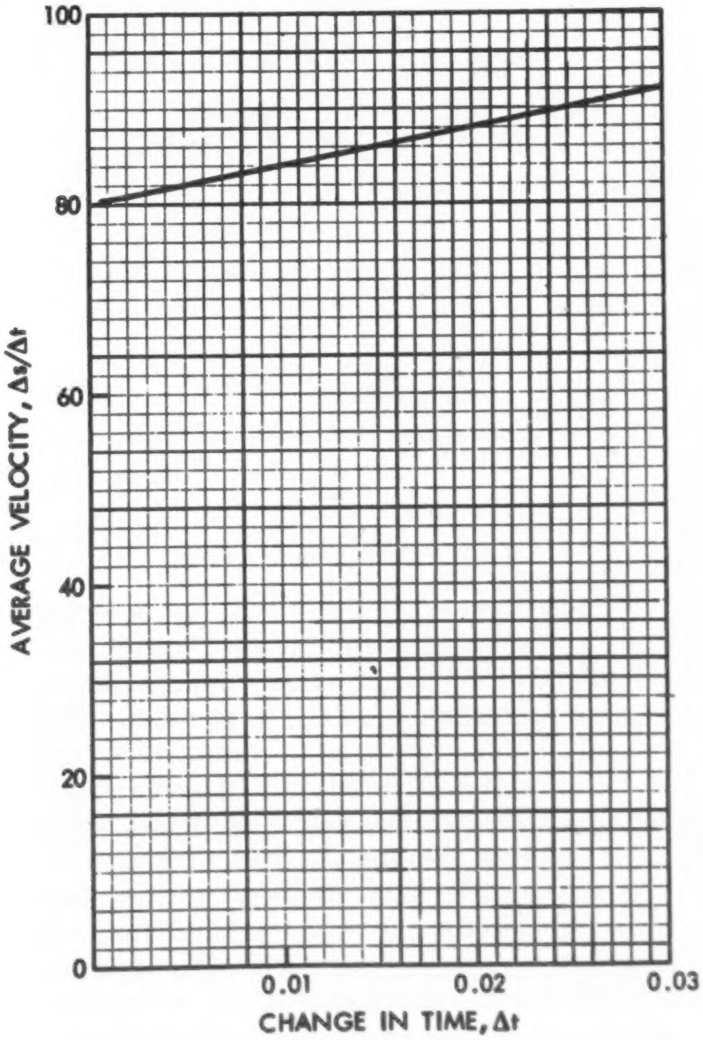


Figure 5-5

Figure 5-5 shows graphically the student's third conclusion. The change in average velocity as Δt approaches zero is linear. The left end of the graph comes closer and closer to 80 as Δt diminishes but it never reaches 80. As you may have already guessed, the number 80 is the velocity of the car at exactly 0.1 seconds from rest!

The *instantaneous velocity* of a body is its velocity at some instant of time. This velocity cannot be computed exactly with the average velocity formula, $\overrightarrow{\Delta s}/\Delta t$, because there is no change in displacement Δs and there is no change in time Δt at an instant of time. Instantaneous velocity is the limit of the average velocity as Δt approaches zero and may be visualized as in Table 5-4. The formal procedure used to find this limit is treated in detail in textbooks on advanced mathematics.

Instantaneous velocity may be interpreted as the slope of a line. The slope of the secant line through points P_1 and P_5 of Figure 5-4 is 120; that is, the average velocity of the car during the interval of time from 0.1 to 0.2 seconds. The average velocity of the car during the interval of time from 0.1 to 0.17 seconds was found to be 108 cm/sec (Table 5-3). This is the slope of the secant line through points P_1 and P_4. We shall refer to the secant lines through P_1 as "average velocity secants." For each value of Δt there is a point P on the graph and an average velocity secant line P_1P. For any sequence of values of Δt approaching zero, there is a sequence of points P approaching P_1 and a sequence of average velocity secant lines. We may think of a secant line pivoting about the point P_1 as Δt decreases. There is a position the average velocity secant can approach but cannot assume because a secant must pass through two distinct points. This position is the tangent line at point P_1. The slope of the tangent line at point P_1 is the instantaneous velocity of the car at exactly 0.1 seconds from rest!

In Chapter 4 you learned that the defining concepts for velocity and acceleration take the same form. An average or steady velocity is the time-rate of change in displacement; that is, $\overrightarrow{\Delta s}/\Delta t$. An average or steady acceleration is the time-rate of change in velocity; that is, $\overrightarrow{\Delta v}/\Delta t$. The meaning of instantaneous acceleration will be easy for you to grasp if you associate it with the meaning of instantaneous velocity. This will be left for you to do as an exercise. You may find it profitable to study the graph in Figure 5-4. In your mind, replace the axis of displacements with an axis of velocities.

5-1 *Exercises Velocity and Acceleration*

1. Graph the data the student collected experimentally (Table 5-1). Use this graph to determine how long it took the car, starting from rest, to travel a distance equal to the distance between the rest point and the first impact mark thereafter.
2. Find an equation that essentially fits the graph of Exercise 1.
3. Use the graph obtained in Exercise 1 to find the instantaneous velocity of the car at exactly 0.14 seconds from rest.
4. Compute the average velocity of the car during the time intervals 0-0.05 sec., 0.05-0.10 sec., 0.10-0.15 sec., 0.15-0.20 sec., and 0.20-0.25 seconds (see Table 5-1). Assume that the velocity of the car at the middle of the time interval is equal to the average velocity for the whole interval. This means that the instantaneous velocity of the car at 0.025 seconds is equal to the average velocity of the car during the time interval 0.05-0.10 sec., 0.10-0.15 sec., 0.15-0.20 velocity of the car will be equal to the average velocity during the time interval 0.05-0.10 seconds. Make a velocity-time graph and read the acceleration of the car from the graph. Save your graph for a later problem.

5-2 More about Kinematics

In Section 4-3 you learned that the formula $\overrightarrow{\Delta s} = \vec{\bar{v}}\Delta t$ can be used to compute the displacement of a body when (a) its average velocity is known or (b) it has a constant velocity. This formula will not suffice in a situation where the motion of a body is irregular and its average velocity is unknown. How, then, is the change in displacement computed under such circumstances? A clue to the answer can be found in Figure 5-6.

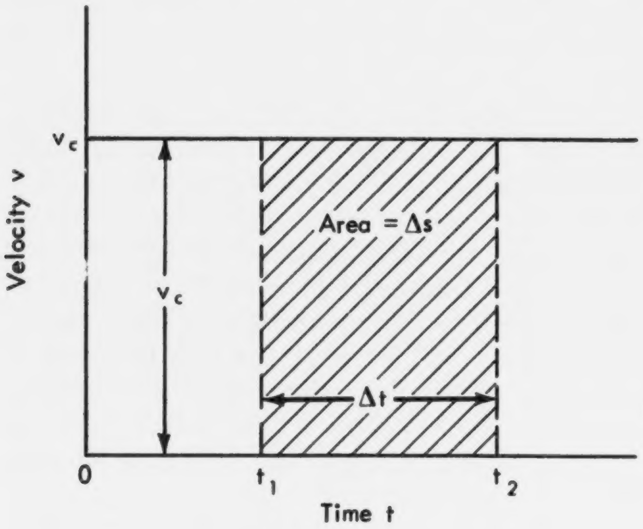


Figure 5-6

This figure is the graph of a body moving with a steady velocity v_c. The area that is bounded by the graph, the time axis, and the lines $t = t_1$ and $t = t_2$ is the change in displacement of the body during the interval of time $t_2 - t_1$.

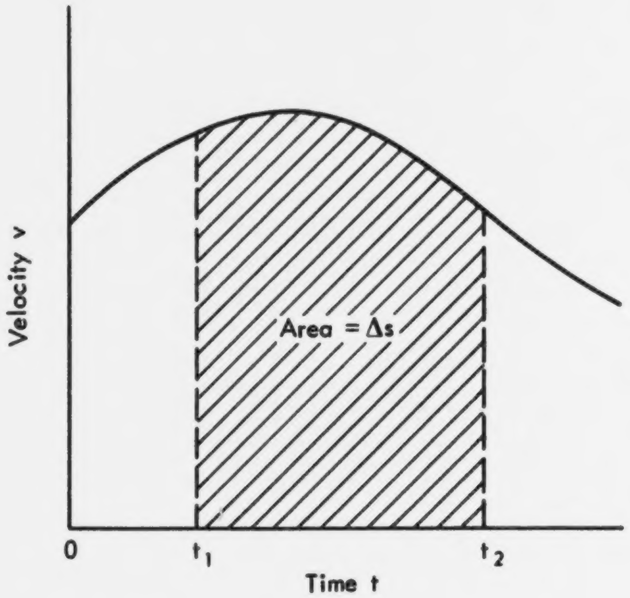


Figure 5-7

Figure 5-7 illustrates a more complex motion. In this case the velocity varies continually and the area has an irregular shape, making computation of the area much more difficult. Techniques used to compute such an area are treated in detail in textbooks of advanced mathematics. Some of these techniques will be discussed in Section 6-5.

If the straight line distance between two towns A and B is 150 miles and 3.25 hours is required to make the trip, the average velocity for the journey is $\overrightarrow{\Delta s}/\Delta t$ with magnitude $150/3.25 = 46.2$ miles/hour. A motion problem of the same type but with a less obvious solution will now be considered. Suppose that the velocity of a body varies according to the equation $v = k + nt^2$ where v is the velocity of the body in m/sec (that is, meters per second) after it has traveled t seconds; k and n are constants whose values are 10 m/sec and 5 m/sec³ respectively. At the end of 3 seconds the velocity of the body

3

will be $10 + 5(3^2)$; that is, 55 m/sec. What will be the average velocity of the body during the first five seconds? A graph of the equation will help you visualize the solution and estimate the answer to the problem. With advanced mathematics the area under the graph, which is equal to the displacement of the body during the first five seconds, can be found. This turns out to be 258 1/3 meters. The average velocity can now be computed with the formula $\overrightarrow{\Delta s}/\Delta t$. The result is 1/5 of 258 1/3; that is, 51 2/3 m/sec. A knowledge of how to compute the average velocity of a uniformly accelerated body will be needed to derive other equations you will use. Hence a formula for this type of problem will be derived.

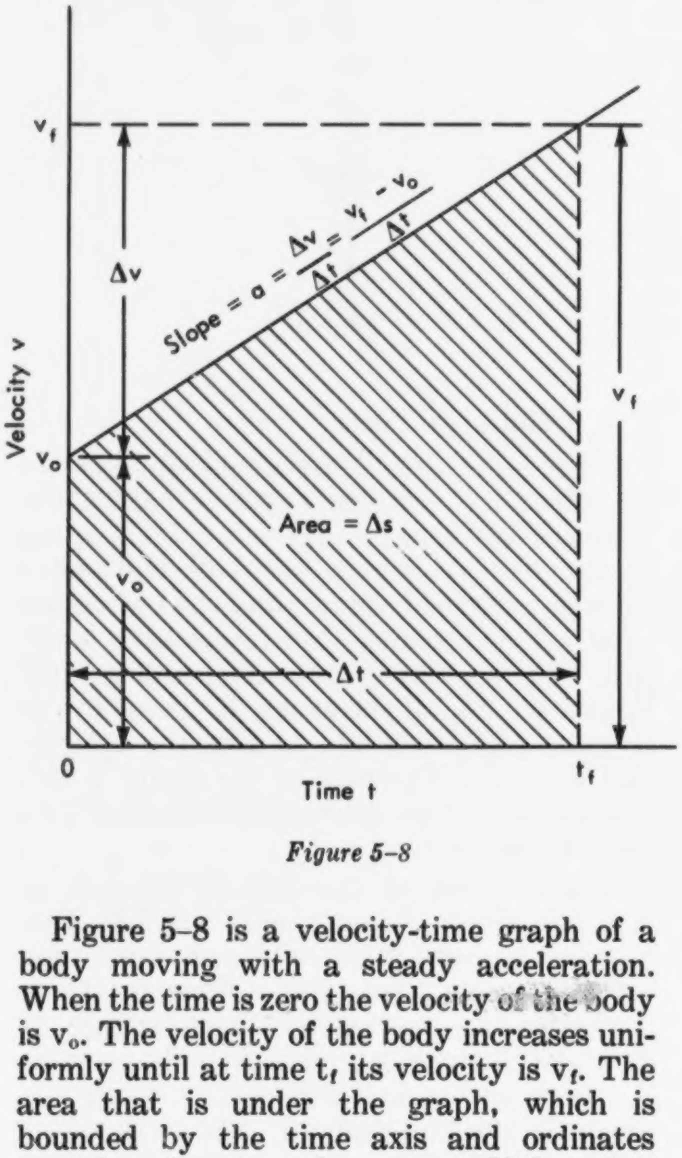


Figure 5–8

Figure 5–8 is a velocity-time graph of a body moving with a steady acceleration. When the time is zero the velocity of the body is v_o. The velocity of the body increases uniformly until at time t_f its velocity is v_f. The area that is under the graph, which is bounded by the time axis and ordinates $t = 0$ and $t = t_f$, is a trapezoidal region. This area can be computed using the formula $A = (1/2)(b + B)h$ where b is the length v_o, B is the length v_f, and h is the altitude Δt.

$$\text{Area} = \Delta s = (1/2)\ (v_o + v_f)\Delta t$$

The magnitude of an average velocity can be computed using the formula $\bar{v} = \Delta s/\Delta t$;

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{(1/2)(v_o + v_f)\Delta t}{\Delta t} = \frac{v_o + v_f}{2}$$

We define

$$\vec{\bar{v}} = \frac{\vec{v}_o + \vec{v}_f}{2}.$$

Consider the problem of a car moving from rest with a uniform acceleration of 4 mi/hr/sec for a period of 20 seconds. What speed will the car attain and how far will it travel during the period of acceleration? In Section 4–4 you learned that average or uniform acceleration is computed using the formula $\vec{a} = \overrightarrow{\Delta v}/\Delta t$. The change in velocity Δv during the period of time Δt may be expressed as $v_f - v_o$ where v_o is the velocity of the body at the beginning of the period of time and v_f is the velocity of the body at the end of the period of time. Hence

$$a = \frac{v_f - v_o}{\Delta t}$$

and $$v_f = v_o + a\Delta t$$

The last equation tells how to compute the speed of the car at the end of the period of acceleration. Since v_o is zero,

$$v_f = a\Delta t = 4 \times 20 = 80 \text{ mi/hr.}$$

The average velocity $\bar{v}$ during the period of acceleration equals

$$\frac{v_o + v_f}{2} = \frac{0 + 80}{2} = 40 \text{ mi/hr.}$$

and the distance the car traveled during the period of acceleration equals $\bar{v}\Delta t = 40 \times 20 = 800$ feet.

Computing the distance the car traveled was really a three step problem. These steps are usually combined into one expression for convenience. This will be done with the aid of Figure 5–8. The change in displacement of the body is expressed as

$$\text{area} = \Delta s = (1/2)(v_o + v_f)\Delta t.$$

A velocity-time graph of a uniformly accelerated body is a straight line. The slope of this line is the acceleration. Hence the acceleration of the body in this problem should be expressed as

$$a = \frac{\Delta v}{\Delta t} = \frac{v_f - v_o}{\Delta t}$$

Solving the acceleration expression for v_f and substituting its value in the displacement equation gives

$$\Delta s = (1/2)(v_o + v_o + a\Delta t)\Delta t$$
$$\Delta s = v_o\Delta t + (1/2)a\,(\Delta t)^2$$

The last equation is commonly found in physics books in the form

$$s = v_ot + (1/2)(at^2)$$

where s is the change in displacement and t is the length of time the body is accelerated. In vector form the equation reads

$$\vec{s} = \vec{v}_ot + (1/2)\vec{a}t^2$$

When a body accelerates from rest the initial velocity v_o is zero. In such cases, the displacement equation for uniform acceleration reduces to

$$\vec{s} = (1/2)\vec{a}t^2$$

5–2 *Exercises Move about Kinematics*

1. Estimate the area under the graph of Exercise 4, Section 5–1, that is bounded by the graph, the time axis, and ordinates $t = 0$ and $t = 0.25$. How does this area compare with the change in displacement of the car during that period of time? (See Table 5–1.)
2. A steady force slows a vehicle at the rate of 10 ft/sec each second. If the force is applied at the instant the vehicle has a velocity of 88 ft/sec (60 mi/hr), find (a) the time required to reduce the velocity of the vehicle to 22 ft/sec (15 mi/hr) and (b) the distance the vehicle travels while undergoing the velocity change. (Hint: A decrease in velocity is a deceleration, or negative acceleration.)
3. Near the surface of Earth, all objects falling freely accelerate downward approximately 32 ft/sec², or 1 g (if the friction drag due to the air is neglected). When an object is projected upward it is decelerated 32 ft/sec², or 1 g. Calculate (a) the distances a body will fall, starting from rest, in 1 second, 3 seconds, and 5 seconds and (b) the velocity the body will have at each instant of time. (Neglect friction due to the air.)
4. An object is projected vertically upward with a velocity of 96 ft/sec. (a) How long will it take the object to reach its peak? (b) How high will it rise? (c) When will the object be 80 feet above the ground?
5. Derive the formula $v_f^2 = v_o^2 + 2as$ from the formulas $v_f = v_o + at$ and $s = v_ot + (1/2)at^2$.

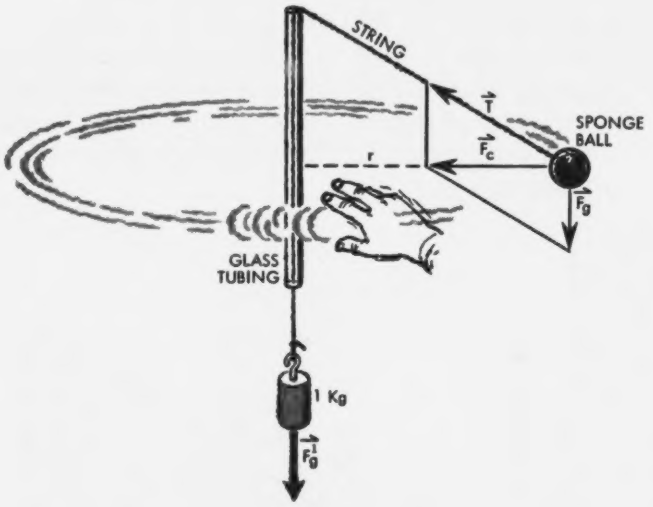


Figure 5–9

5–3 Centripetal Force

A feel for a *central,* or centripetal, *force* can be gotten by whirling a sponge ball in the manner illustrated by Figure 5–9. If the speed of the ball is correct the ball will move through a circle of radius r and the 1 kilogram mass will not move. Two forces act on the ball. They are the downward force of gravity $\vec{F}_g$ and the pull of the string $\vec{T}$. These forces (vector quantities) add to give the centripetal force $\vec{F}_c$. The pull of the string $\vec{T}$ is a consequence of the gravitational force $\vec{F}'_g$ on the 1 kilogram mass.

If the speed of the ball is allowed to diminish, the 1 kilogram mass will fall. This is an indication that the resultant force on the ball is greater than the centripetal force needed to keep the ball in its circular path. If the speed of the ball is increased, the centripetal force on the ball will not be enough to constrain the ball and the 1 kilogram mass will be raised. The centripetal force required to hold a mass in a specific circular path is critical. Centripetal force $\vec{F}_c$ is an unbalanced force and, according to Newton's second law of motion, should accelerate the sponge ball in the direction of $\vec{F}_c$. An acceleration caused by a centripetal force is referred to as *centripetal acceleration.* Centripetal acceleration will be explained with the aid of Figures 5–10, 5–11, and 5–12.

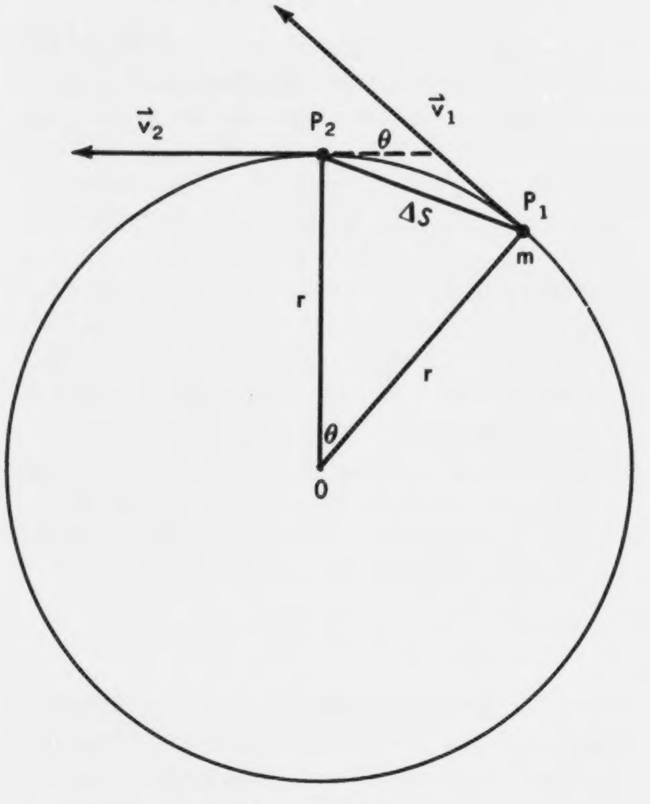


Figure 5–10

Mass m of Figure 5–10 is moving through a circle of radius r with a steady speed. Its instantaneous velocity at point P_1 is $\vec{v}_1$. The direction of this velocity is the direction the mass would move, at that instant, if the circular motion ceased. During a small interval of time Δt, the mass goes through an angular displacement θ and arrives at point P_2 with an instantaneous velocity $\vec{v}_2$. The average velocity of the mass during this interval of time is $\overrightarrow{\Delta s}/\Delta t$. Velocities $\vec{v}_1$ and $\vec{v}_2$ have the same magnitude because the speed of the mass does not change. A change in the direction of a velocity like a change in the magnitude of a velocity signifies an acceleration (Chapter 4). Hence, the ever changing direction of the mass m as it moves through the circle signifies a continuous acceleration of the mass m.

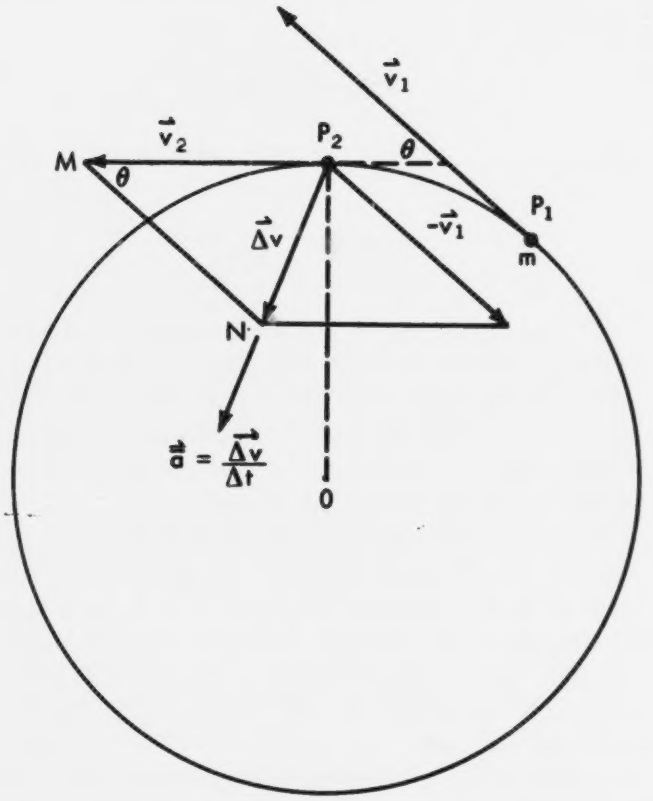

$\vec{v}_1$
$\vec{v}_2$
P_2
θ
M
θ
$\overrightarrow{\Delta v}$
$-\vec{v}_1$
P_1
m
N
$\vec{a} = \frac{\overrightarrow{\Delta v}}{\Delta t}$
0

Figure 5–11

The change in velocity of mass m as it goes from point P_1 to point P_2 is found by subtracting $\vec{v}_1$, its instantaneous velocity at point P_1, from $\vec{v}_2$, its instantaneous velocity at point P_2. This is accomplished by adding the negative of $\vec{v}_1$ to $\vec{v}_2$. Figure 5–11 shows the subtraction. The change in velocity $\overrightarrow{\Delta v}$ divided by Δt gives the average acceleration of mass m as it moves from point P_1 to point P_2.

Notice that the direction of the average acceleration is not toward the center 0 of the circle. Angle OP_2M is a right angle; $\angle NP_2M$ is a base angle of isoscoles triangle NP_2M with vertex angle M equal to θ. Therefore $\angle NP_2M = (1/2)(180° - \theta) = 90° - (1/2)\theta$ and $\angle OP_2N = (1/2)\theta$. Thus as the angular displacement θ approaches zero, the average acceleration $\overrightarrow{P_2N}/\Delta t$ approaches a position along $\overrightarrow{P_2O}$.

Triangle P_1OP_2 of Figure 5–10 is similar to triangle P_2MN of Figure 5–11 since both triangles are isosceles triangles with a vertex angle θ. Therefore,

$$\frac{\overline{P_2N}}{\overline{P_1P_2}} = \frac{\overline{P_2M}}{\overline{P_1O}};$$

$$\frac{\Delta v}{\Delta s} = \frac{v}{r}$$

where v is the magnitude of the instantaneous velocity. Solving the expression for Δv results in

$$\Delta v = \left(\frac{v}{r}\right)\Delta s.$$

Dividing both sides of the equation by Δt gives

$$\frac{\Delta v}{\Delta t} = \frac{v}{r} \times \frac{\Delta s}{\Delta t}$$

The ratio $\Delta s/\Delta t$ in the last equation is the average velocity of the mass as it moves from point P_1 to point P_2. The ratio $\Delta v/\Delta t$ is the average acceleration of the mass as it moves from point P_1 to point P_2. The ratio v/r is a constant in the equation. The first two statements concerning the last equation give a picture of average motion. To obtain a picture of instantaneous motion, it is necessary to evaluate the average velocity ($\Delta s/\Delta t$) and the average acceleration ($\Delta v/\Delta t$) as Δt diminishes and approaches zero as a limit. As Δt approaches zero as a limit the average velocity will approach a limit that we call the *instantaneous velocity*. As Δt approaches zero as a limit the average acceleration will approach a limit that we call the *instantaneous acceleration*. Substituting the instantaneous values in the last equation for $\Delta v/\Delta t$ and $\Delta s/\Delta t$ results in the following formula for the magnitude of the centripetal acceleration

$$a = \frac{v^2}{r}.$$

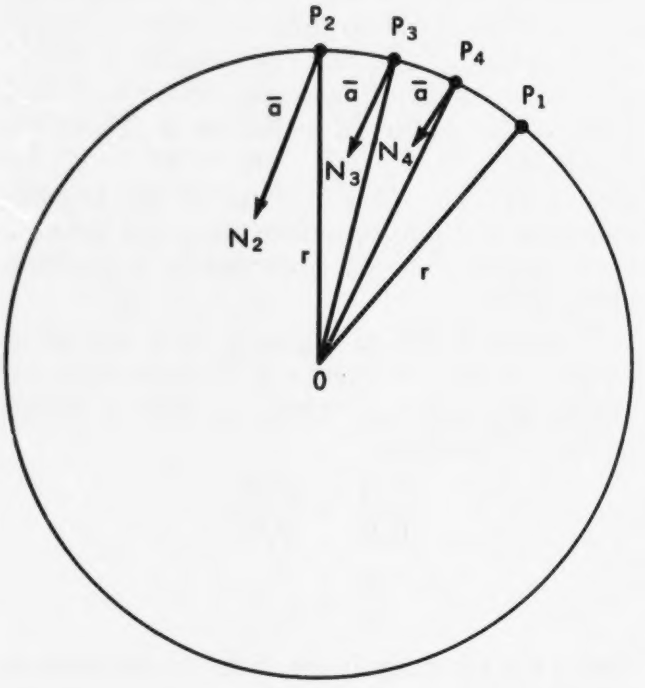


Figure 5–12

Figure 5–12 will help you visualize the direction of centripetal acceleration. As in Figure 5–11,

$$\angle OP_2N_2 = (1/2)\angle P_1OP_2 = (1/2)\theta$$
$$\angle OP_3N_3 = (1/2)\angle P_1OP_3$$
$$\angle OP_4N_4 = (1/2)\angle P_1OP_4$$

The average acceleration vectors $\overrightarrow{P_2N_2}$, $\overrightarrow{P_3N_3}$, and $\overline{P_4N_4}$ indicate the direction and relative magnitude of the average acceleration associated with the movement of mass m from point P_1 to P_2, P_3, and P_4. The closer the point is to P_1, the smaller the angular displacement and the closer the average acceleration vector comes to being directed toward the center O. At the limit the acceleration is instantaneous and radial (that is, directed toward the center of the circle).

An expression for the centripetal force required to hold a mass in a circular path may be obtained by substituting the value of centripetal acceleration (v^2/r) for a in Newton's second law formula, $F = ma$. Then

$$F = \frac{mv^2}{r}$$

where F is the centripetal force in newton's (Section 5–4), m is the mass in kilograms, v is the velocity of the mass in meters/second, and r is the radius of the circle in meters.

5–3 *Exercises Centripetal Force*

1. A metal ball whose mass is 2 kilograms and whose weight is 19.6 newtons (about 4.4 lbs.) is fastened to one end of a string 1 meter long. The other end of the string is fastened to a fixed point P on an overhead support. The ball describes a horizontal circle whose center is directly under P and the string makes an angle of 30° with the vertical. (a) Draw a diagram of the problem. (b) Show the forces that act on the ball. (c) Compute the centripetal force, and (d) calculate the velocity of the ball.
2. A solid object whose mass is 1 kilogram and whose weight is 9.8 newtons is placed in a bucket and the bucket is whirled through a circle in a vertical plane. If the radius of the circle is 0.8 meters, what is the least velocity the object can have at the top of the path and not fall out of the bucket?

5-4 Circular Orbits

Measurements are usually made by physicists in a system of units called the *absolute system*. In this system, all mechanical notions are defined in terms of three fundamental concepts. These are length, mass, and time. When length is measured in meters, mass is measured in kilograms, and time is measured in seconds.

Force is a concept whose unit of measure, in the absolute system, is derived from Newton's second law of motion, $F = ma$; 1 unit of force (*newton*) is that force which will cause 1 unit of mass (kilogram, abbreviated kg) to be accelerated 1 unit (m/sec^2).

The mechanics of several topics to follow will require the use of a special force called weight, so it is essential that you know how the physicist expresses weight. The physicist expresses *weight* as mg, or the product of mass and its acceleration due to gravity. Let us see why this is done. The formula $F = ma$ reads as follows when used with absolute units.

> The accelerating force in absolute units is equal to the product of the mass in the absolute system and the acceleration

When this formula is applied to a falling body; the weight force W is the accelerating force F, the mass of the body is m, and the acceleration of the body is g. Upon substituting these facts in the displayed statement you obtain

> The weight force in absolute units is equal to the product of the mass in the absolute system and g;

$$W_{newtons} = m_{kg} g_{m/sec^2}.$$

The value of g at the surface of Earth is approximately 9.8 meters/sec². Thus a 10 kilogram mass weighs approximately 98 newtons at the surface of Earth.

In 1686 Sir Isaac Newton announced what is now known as Newton's *law of universal gravitation*. This law is stated as follows: Each particle of matter attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. In mathematical form the law reads

$$F \alpha \frac{mM}{R^2}$$

where F is the force of attraction between two particles whose masses are m and M, and R is the distance between them.

A spherical body responds to a gravitational force as though its mass is concentrated at a point called the *center of mass*. This point is the center of a sphere if the mass is uniformly distributed. The distance R between two spheres is the distance between their centers of mass.

Newton's law of gravitation as stated is unsatisfactory for making quantitative predictions because it is a proportionality and not an equation. The proportionality does show that the ratio

$$\frac{F}{\frac{mM}{R^2}}$$

must be a constant G and thus

$$F = G \times \frac{mM}{R^2}$$

This constant G, which is now known as the *gravitational constant*, has been found experimentally by measuring the gravitational force between known masses that are a known distance apart. This was first done by Sir Henry Cavendish in 1789. The value of the gravitational constant G as found by direct measurement is 6.670×10^{-11} $\frac{\text{newton (m)}}{\text{kg}^2}$. When stated with G the formula for Newton's law of gravitation is in a form suitable for use with the absolute units introduced.

Next consider two particles whose masses are m and M. When the particles are at a distance R_1,

$$F_1 = G\frac{mM}{R_1^2}$$

When the particles are at a distance R_2,

$$F_2 = G\frac{mM}{R_2^2}$$

Thus Newton's law of gravitation enables us to express the relationship between gravitational force and the distance between two particles as a proportion;

$$\frac{F_1}{F_2} = \frac{R_2^2}{R_1^2}$$

where F_1 is the force when the distance between the particles is R_1 and F_2 is the force when the distance is R_2.

Table 5-5

Relative Distance Between Masses	Relative Force of Attraction
1	1
2	¼
10	$\frac{1}{100}$
100	$\frac{1}{10{,}000}$
1,000	$\frac{1}{1{,}000{,}000}$

Table 5-5 shows how the gravitational force decreases as two particles separate.

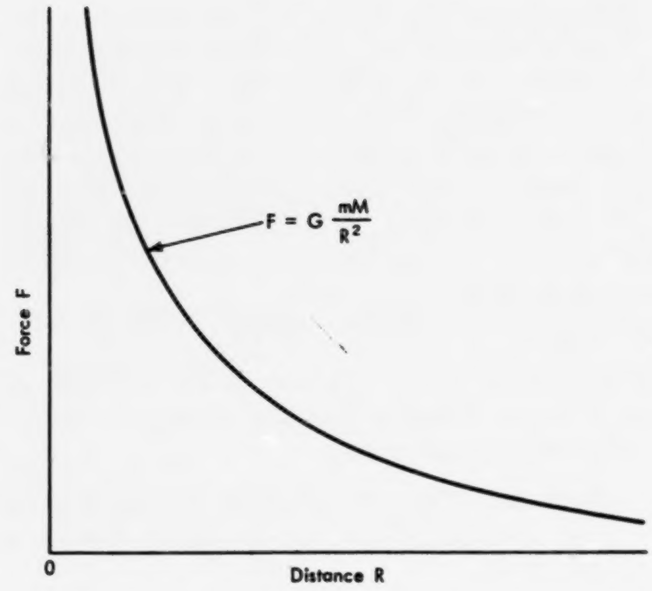


Figure 5-13

Figure 5-13 shows graphically how the gravitational force *F* changes as the distance *R* between two particles changes. The graph is asymptotic to each of the axes; that is, it approaches but never meets either axis. Hence, a position of zero gravitational force is impossible.

The small letter *g* is the symbol for acceleration due to gravity and it should not be confused with capital *G*, the symbol for the gravitational constant. Acceleration due to gravity is not a constant but its value is influenced by a number of factors. Of these factors, the distance from the center of Earth and the surface speed of Earth are responsible for the greatest variations. We will consider the effect of distance. When a mass *m* is at a distance *R* from the center of Earth it experiences a force *F* equal to

$$G\frac{mM}{R^2}$$

This force is the weight force, *mg*. Hence

$$mg = G\frac{mM}{R^2}$$

$$g = \frac{GM}{R^2}$$

This equation shows that *g* is inversely proportional to the square of the distance between the center of Earth and the point where *g* is measured. For two different distances R_1 and R_2 we have

$$g_1 = \frac{GM}{R_1^2}$$

$$g_2 = \frac{GM}{R_2^2}$$

$$\frac{g_1}{g_2} = \frac{R_2^2}{R_1^2}$$

where g_1 is the value of *g* at a distance R_1 from the center of Earth and g_2 is the value at a distance R_2.

When an artificial satellite is in a circular orbit around Earth, the contripetal force required to keep the satellite in the orbit is the gravitational force between the satellite and Earth. The velocity it must have to be launched into its orbit can be found by equating the centripetal force with the gravitational force;

$$\frac{mv^2}{R} = G\frac{mM}{R^2}$$

$$v = \sqrt{\frac{GM}{R}}$$

where v is the circular orbital velocity of a satellite in m/sec., *M* is the mass of the earth in kilograms, *R* is the radius of the orbit in meters, and *G* is the gravitational constant in $\frac{\text{newton (m)}}{\text{kg}^2}$.

Weightlessness is a word that has become a part of our vocabulary in recent years and is certainly one that is often misunderstood. The common belief that an astronaunt becomes weightless because gravity ceases to act on him is absurd. A little experimenting with Newton's law of gravitation will verify that statement. The nature of weightlessness was spelled out in a subtle way when the formula for the velocity of a satellite in circular orbit was derived. If you will review the derivation you will note that *m*, the mass of the satellite, occurred in both members of the first equation and did not occur in the expression for v.

This means that the velocity needed to place a mass in a given circular orbit is independent of the mass! The many free masses that are a part of a spacecraft are themselves in circular orbits. They are held there by the force of gravity. Astronaut John Glenn aptly described this state of affairs when he pointed out that he soon learned to let go of his pen where it was rather than lay it down. Gravity, or centripetal force, will not move a mass nearer to the center of Earth when the mass is traveling in a circular orbit. Under such a condition, one free mass in a spacecraft would not exert a "down" force on another. This condition is known as weightlessness.

During the third orbit of NASA's Gemini-4 spacecraft astronaut Edward H. White II became the first American astronaut to leave his spacecraft while in orbit. He demonstrated in a most spectacular manner the condition of weightlessness and the fact that he himself was in orbit. White was secured to his spacecraft by a 25-foot umbilical line and a 23-foot tether line, both wrapped together with gold tape to form one cord. Because of his weightlessness astronaut White used a hand-held self-maneuvering unit to move about. He remained outside the spacecraft for a total of 21 minutes.

5–4 *Exercises Circular Orbits*

1. The diameter of Earth is approximately 8,000 miles. At what distance from the surface of Earth is your weight only ¾ of what it is at the surface?

2. What is the value of *g* at an altitude of one Earth's radius? Assume the value of *g* at the surface to be 32 ft./sec.^{2}

3. Compute the mass of Earth in kilograms. $G = 6.670 \times 10^{-11} \frac{\text{newton (m)}}{\text{kg}^2}$, the radius of Earth is 6.37×10^6 meters, and $g = 9.8$ m/sec^2 at the surface of Earth. Hint: The gravitational force *F* and the weight force *mg* are the same force.

4. The mass of the moon is about 1/80 of the mass of Earth. The distance between their centers varies from about 252,972 miles to 221,614 miles. At what point between Earth and the moon is the gravitational force of Earth balanced by the gravitational force of the moon when they are 240,000 miles apart?

5. Show that the velocity required to place a satellite in circular orbit may be expressed as $\sqrt{gR}$ where *R* is the radius of the orbit and *g* is the acceleration due to gravity at the elevation of the orbit.

6. (a) Compute the velocity required to place a spacecraft in a circular equatorial orbit around Earth at an altitude of 4.83×10^5 meters (about 300 miles). Consider the mass of Earth to be 6×10^{24} kilograms (about 1.23×10^{25} pounds) and the equatorial radius of Earth to be 6.37×10^6 meters.

(b) What velocity would be required for a similar orbit around the moon? The mass of the moon is 1/80 the mass of Earth and the radius of the moon is 1.740×10^6 meters (about 1081 miles).

7. An experimenter measured the value of *g* by measuring the acceleration of a car as it coasted down each of several wires inclined with respect to the horizontal. The following data was collected:

Acceleration in ft/sec^2	Angle of Inclination
3.15	15°
6.08	30°
9.31	50°
11.43	70°

(a) Make a graph of the acceleration and the sine of the angle.

(b) Use your graph to determine the value of *g*.

(c) Was the experimenter on Earth, Mars, or the moon? The value of *g* at the surface of Earth is 32 ft/sec^2, on the surface of Mars it is 0.38 of what it is on Earth, and on the surface of the moon it is 1/6 of what it is on Earth.

5–5 The Earth-Synchronous Satellite

The period of a satellite is the time required for a satellite to move once through its orbit. If the orbit is circular with radius R, then the period T is simply the distance $2\pi R$ around the orbit divided by the velocity v;

$$T = \frac{2\pi R}{v}$$

When $\sqrt{GM/R}$, the velocity of a satellite in circular orbit, is substituted for V, we have

$$T^2 = \frac{4\pi^2 R^3}{GM}$$

$$T = 2\pi\sqrt{\frac{R^3}{GM}}$$

where T is the period of the satellite in seconds, R is the radius of the orbit in meters, M is the mass of the earth in kilograms, and G is the gravitational constant in · newton (m)/kg^2.

An Earth-synchronous satellite is a satellite whose orbital motion is synchronized with the rotation of Earth. The satellite has a circular orbit and a 24 hour period. If the orbit of the satellite is in the equatorial plane, then the position of the satellite relative to a fixed point on Earth is fixed; that is, an Earth-based observer would always see the satellite over the same spot on earth. If the orbit is inclined to the equator, the position of the satellite will not appear to an observer on Earth to be fixed. Rather the satellite will appear to oscillate north and south in a figure-eight pattern but stay close to the same longitude.

The radius of the orbit required to achieve synchronization can be found by solving the equation

$$T^2 = \frac{4\pi^2 R^3}{GM}$$

for R;

$$R = \sqrt[3]{\frac{T^2 GM}{4\pi^2}}$$

For an orbit in the equatorial plane T = $24 \times 60 \times 60 = 86{,}400$ and

$$R \approx \sqrt[3]{\frac{(86{,}400)^2\,(6.67 \times 10^{-11})\,(6 \times 10^{24})}{4(3.1416)^2}}$$

$$\approx \sqrt[3]{75.62 \times 10^{21}}$$

This value for R, the distance from the center of Earth to the satellite, is about 4.229×10^7 meters; that is, about 26,270 miles. Since the equatorial radius of Earth is 3,963 miles, an Earth-synchronous satellite in the equatorial plane must orbit at an altitude of about 22,307 miles. The velocity required to place the satellite in the synchronous orbit is computed as follows:

$$v = \sqrt{\frac{GM}{R}} \approx \sqrt{\frac{(6.67 \times 10^{-11})\,(6 \times 10^{24})}{4.229 \times 10^7}}$$

$$\approx 3076 \text{ m/sec.}$$

A velocity of 3076 m/sec is equivalent to 10,089 ft/sec and to 6879 mi/hr.

Communication channels in the United States have been increased considerably in recent years by the use of microwave radio transmission. Vast distances are linked by microwave relay stations. These stations receive signals and, after suitable amplification, pass them along a line of sight distance, or from horizon to horizon. Line of sight transmission is one factor that limits the distance between stations, so microwave relay stations are located on high towers or mountains.

There is an increasing need for world-wide communication channels so microwave radio transmission is being experimented with for this purpose. The principle is simple. Intelligence from a communications system is fed into a powerful transmitter. It sends a line of sight microwave signal to a satellite in orbit. After suitable amplification, the satellite passes along a line of sight transmission to a receiving station several thousand miles away. The output of the receiving station is fed into the receiving communications network.

On July 26, 1963, NASA launched Syncom II, an Earth-synchronized communications satellite. It was placed in an elliptical trajectory and carried to an altitude of about 22,300 miles by a three-stage Delta rocket. Syncom II was then propelled into a circular orbit by a propulsion unit of its own called an "apogee kick" rocket motor. The satellite was properly positioned in its orbit for radio relay duty by ground command using the thrust of hydrogen peroxide jets. The orbit of Syncom II was inclined 30° to the equator so its position to an observer on Earth was not fixed. It stayed close to the same longitude but moved north and south in a figure-eight pattern.

On August 19, 1964, NASA placed Syncom III in a true equatorial Earth-synchronous orbit. This satellite's position relative to Earth remains fixed. Three such satellites spaced 120 degrees apart can cover all areas of the world by line of sight radio transmission except a small portion of each polar region.

5-5 *Exercises The Earth-Synchronous Satellite*

Use these values in the exercises:

$G = 6.670 \times 10^{-11}$ newton (m)/kg^2
Radius of Earth = 6.37×10^6 meters
Mass of Earth = 6×10^{24} kilograms
Mass of moon = 1/80 mass of Earth
Radius of the moon = 1.740×10^6 meters
Period of the moon = 29 days, 12 hours, 44 minutes, and 3 seconds

1. Calculate (a) the size of a moon-synchronous orbit and (b) the velocity required for a spacecraft to be in such an orbit.

2. Echo I was placed in an elliptical orbit with a perigee of 812.1 miles and an apogee of 906.5 miles. Its period was 118.3 minutes. What period would Echo I have had if the desired circular orbit at 900 miles (1.449×10^6 meters) altitude had been realized?

3. Kepler's first law for planetary motion states that the orbit of each planet is an ellipse with the sun at one of its foci. Kepler's third law states that the squares of the periods of the planets are proportional to the cubes of their mean distances from the sun. Use Newton's law of gravitation

$$F = G\frac{mM}{R^2}$$

to show that Kepler's third law proportion also holds true for circular orbits.

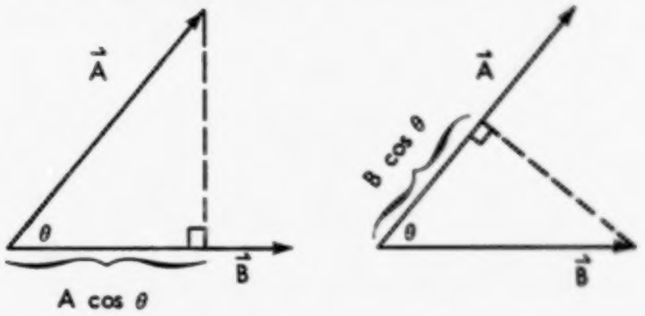
5-6 The Escape Velocity

Getting a spacecraft away from the Earth involves work and the imparting of energy to the spacecraft. This section will develop the concepts of work and energy as they pertain to the problem of determining the altitude to which a satellite will rise after launch.

The *scalar product* of two vectors is defined as the product of their magnitudes and the cosine of the angle formed by them. The scalar product of vectors $\vec{A}$ and $\vec{B}$ is expressed as

$$\vec{A} \cdot \vec{B} = AB \cos \theta$$

and is read "A dot B equals AB cosine theta" where A and B are the magnitudes of vectors $\vec{A}$ and $\vec{B}$ respectively and θ is the angle formed by them. When you find the scalar product of vectors $\vec{A}$ and $\vec{B}$ you are finding the product of the magnitude of one vector (either one) and the projection of the other vector along the first vector (Figure 5-14).



$\vec{A} \cdot \vec{B} = B\ (A \cos \theta)$ $\vec{A} \cdot \vec{B} = A\ (B \cos \theta)$

Figure 5-14

Work is done on a mass when a force causes the mass to be displaced. Computing work is an example of finding a scalar product. *Work*, a scalar quantity, is defined as the product of force and displacement, two vector quantities.

$$W = \vec{F} \cdot \vec{S} = FS \cos \theta$$

where W is the work done in joules, $\vec{F}$ is the steady or average force in newtons, and $\vec{S}$ is the displacement in meters. Observe that FS cos θ reduces to FS when the angle formed by the vectors is 0°. Also note that FS cos $\theta = 0$ when the angle formed by the vectors is 90°.

The work problems with which we will be concerned are those where a force $\vec{F}$ acts along the line of displacement $\vec{s}$ and angle θ is zero. The force may or may not be steady.

If the force is known to be steady or is the average of a variable force, the work can be computed using the formula W = FS. If the force is variable and its average is unknown, the work must be computed by other means.

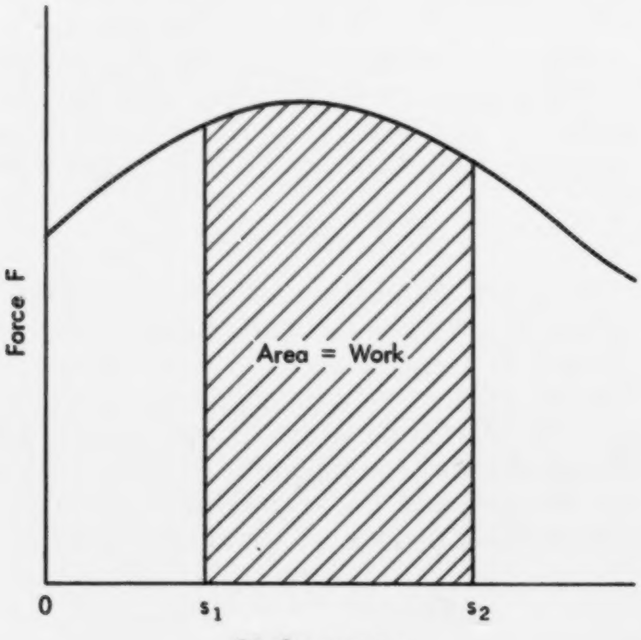


Figure 5-15

Think of a rectangle whose width is expressed in units of displacement, whose length is expressed in units of force, and whose area is expressed in units of work. Figure 5-15 is a force-displacement graph. It shows how the force used to propel a mass varied during the displacement. The work done during the change in displacement $s_2 - s_1$ is measured by the area which is bounded by the graph of the force, the displacement axis, and the lines $s = s_1$ and $s = s_2$.

Energy is the capacity for doing work. A stretched spring, for example, is said to possess energy because it can raise a mass when the lower end of the spring is released. A falling pile hammer is said to possess energy because it does work on the pile as it is stopped. The quantity of energy possessed by a body is equal to the amount of work that can be derived from it. It is almost needless to say that work and energy are measured in the same units. Just as energy gives rise to work; work gives rise to energy. The amount of energy imparted to a mass is equal to the amount of work done on it.

Raising a mass to a higher elevation requires work so the energy of the mass will be increased by the amount of work done. Energy derived in this fashion is called *gravitational potential energy*. If a body of mass m is moved a short distance with a steady speed from the surface of Earth to an elevation h, the force required will be its weight force mg and the work done will be mgh. The gravitational potential energy of mass m with reference to the surface of Earth should, therefore, be expressed as

$$E_P = mgh$$

where E_P is the gravitational potential energy in joules, m is the mass in kilograms, h is the vertical displacement in meters and g is 9.8 m/sec^2.

One assumption was made in arriving at the statement $E_P = mgh$. It was assumed that g would be constant for the problem. This assumption is acceptable only when the vertical displacement h is small. If h is a large displacement into space, the value of g will change significantly so $E_P = mgh$ will not be a true statement of the gravitational potential energy of mass m. How is the work and hence the gravitational potential energy of a mass computed when it is raised to a height of several hundred miles above Earth? This will be explained with the aid of Figure 5-16.

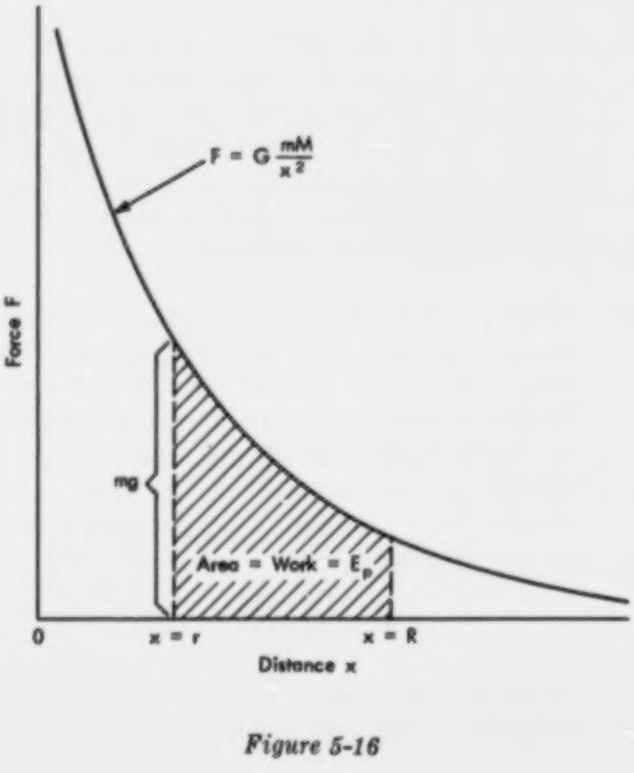


Figure 5-16

Consider a small mass m at a distance x from the center of Earth (mass M). Let R be the distance from the center of Earth to some position in space. The gravitational force $\vec{F}$ and therefore the force required to move the mass m from the surface of Earth ($x = r$) to the position in space ($x = R$) decreases steadily as x increases. In other words, at $x = r$ this required force is $\overrightarrow{mg}$ where $g = 9.8$ m/sec.^{2} As x increases from r to R, g decreases and the force $\overrightarrow{mg}$ decreases as shown in the graph in Figure 5–16. The work required to move the mass m from the surface of Earth to a distance R from the center of the Earth is measured by the area that is bounded by this graph, the x-axis, and the lines $x = r$ and $x = R$. We can estimate this area. When the area is computed exactly using the equation

$$F = G\frac{mM}{x^2}$$

for the graph, the area and thus the work W may be expressed by the formula

$$W = GmM\left(\frac{1}{r} - \frac{1}{R}\right).$$

Thus the gravitational potential energy of a mass m at a distance R from the center of Earth is

$$E_P = GmM\left(\frac{1}{r} - \frac{1}{R}\right)$$

where E_P is the potential energy in joules when G, m, M, r, and R are expressed in the absolute units introduced in this chapter.

How much gravitational potential energy does mass m have if R is allowed to increase without bound? Does the energy increase without bound? Consider the expression $GmM\left(\frac{1}{r} - \frac{1}{R}\right)$ as R increases without bound. As R becomes larger and larger, $\frac{1}{R}$ becomes smaller and smaller, and the expression $GmM\left(\frac{1}{r} - \frac{1}{R}\right)$ is getting nearer and nearer to $GmM\left(\frac{1}{r} - 0\right)$. Therefore the potential energy of mass m approaches $\frac{GmM}{r}$!

According to the work-energy principle, if work is done on a mass to increase its velocity, the energy of the mass will increase by the amount of work done. Energy acquired in this fashion is called *kinetic energy.* Imagine a steady force $\vec{F}$ that acts on a body of mass m for a period of time t. This impulse (Chapter 4) causes the momentum of the mass to change by an amount $mv_f - mv_o$ where v_o is the initial velocity of the mass and v_f is its final velocity. According to the work-energy principle

$$\Delta E_k = \text{Work on mass m}$$

$$\Delta E_k = Fs = mas$$

$$s = v_o t + \tfrac{1}{2} at^2$$

$$\Delta E_k = ma(v_o t + \tfrac{1}{2} at^2)$$

$$\Delta E_k = mav_o t + \tfrac{1}{2} ma^2t^2$$

$$a = \frac{v_f - v_o}{t}$$

$$a^2 = \frac{v_f^2 - 2v_f v_o + v_o^2}{t^2}$$

$$\Delta E_k = m\left(\frac{v_f - v_o}{t}\right)v_o t + \tfrac{1}{2} m\left(\frac{v_f^2 - 2v_f v_o + v_o^2}{t^2}\right)t^2$$

$$\Delta E_k = mv_f v_o - mv_o^2 + \tfrac{1}{2} mv_f^2 - mv_f v_o + \tfrac{1}{2} mv_o^2$$

$$\Delta E_k = \tfrac{1}{2} mv_f^2 - \tfrac{1}{2} mv_o^2$$

The last equation shows that mass m had an initial kinetic energy of $\frac{1}{2} mv_o^2$ and a final kinetic energy of $\frac{1}{2} mv_f^2$. The general equation for the kinetic energy of a body is

$$E_k = \tfrac{1}{2}mv^2$$

where E_k is the kinetic energy in joules, m is the mass of the body in kilograms, and v is the velocity of the body in m/sec.

In the absence of outside forces other than gravity the sum of the kinetic and gravitational potential energies of a body is conserved; that is, remains constant. When a body falls freely in a vacuum its loss of gravitational potential energy reappears as an increase in kinetic energy, but its total mechanical energy remains constant. When a body coasts vertically upward (in a vacuum) the body's kinetic energy is gradually transformed to gravitational potential energy, but its total mechanical energy remains constant. Conservation of mechanical energy will be used to predict the velocity a satellite must have to escape from Earth.

A launch vehicle which is composed of three stages carries a satellite of mass m to a distance R_1 from the center of Earth. Suppose that the velocity acquired by the satellite during this journey is v. The total energy of the satellite after launch is the sum of its kinetic and gravitational potential energies. This is expressed by the equation

$$E_{total} = GmM\left(\frac{1}{r} - \frac{1}{R_1}\right) + \tfrac{1}{2}mv^2$$

How high will the satellite go if allowed to coast upward? Like a baseball thrown into the air, the satellite will be decelerated (slowed down) by gravity as it soars upward to its peak at a distance R_2 from the center of Earth. In going from distance R_1 to distance R_2 the satellite's change in potential energy is equal to its change in kinetic energy. The energy of the satellite is conserved. Using this as our starting point, distance R_2 is computed as follows:

$$\Delta E_P = \Delta E_k$$

$$GmM\left(\frac{1}{r} - \frac{1}{R_2}\right) - GmM\left(\frac{1}{r} - \frac{1}{R_1}\right) = \tfrac{1}{2}mv^2 - 0$$

$$\frac{GmM}{r} - \frac{GmM}{R_2} - \frac{GmM}{r} + \frac{GmM}{R_1} = \tfrac{1}{2}mv^2$$

$$\frac{GmM}{R_1} - \frac{GmM}{R_2} = \tfrac{1}{2}mv^2$$

$$2GMR_2 - 2GMR_1 = R_1R_2v^2$$

$$2GMR_2 - R_1R_2v^2 = 2GMR_1$$

$$R_2 = \frac{2GMR_1}{2GM - R_1v^2}$$

$$R_2 = \frac{R_1}{1 - \dfrac{R_1v^2}{2GM}}$$

Observe that the mass of the satellite does not affect the answer. All satellites with a velocity v away from Earth and at a distance R_1 from the center of Earth will coast upward until they are a distance R_2 from the center of Earth.

A satellite increases its gravitational potential energy as it increases its distance from the center of Earth. However, we have seen that the gravitational potential energy of a mass m cannot exceed GmM/r. A satellite will escape from Earths' gravitational field if the satellite's kinetic energy at burnout is enough to increase the potential energy of the satellite to at least Gm M/r. If burnout for a satellite occurs at a distance R from the center of Earth, then the minimum velocity v_e needed for the satellite to escape Earth's gravitational field may be found as follows:

$$\Delta E_P = \Delta E_k$$

$$G\frac{mM}{r} - \left[GmM\left(\frac{1}{r} - \frac{1}{R}\right)\right] = \tfrac{1}{2}mv_e^2 - 0$$

$$G\frac{mM}{r} - G\frac{mM}{r} + G\frac{mM}{R} = \tfrac{1}{2}mv_e^2$$

$$\frac{GmM}{R} = \tfrac{1}{2}mv_e^2$$

$$v_e = \sqrt{\frac{2GM}{R}}$$

Observe that the escape velocity at a given distance R from the center of Earth is the same for all satellites. Also note that the escape velocity diminishes as distance R increases.

5-6 *Exercises The Escape Velocity*

Use these values in the exercises:

$G = 6.670 \times 10^{-11}$ newton (m)/kg^2
Radius of Earth = 6.37×10^6 meters
Mass of Earth = 6×10^{24} kilograms

1. A spacecraft of mass 8800 kilograms (9.68 tons) is in a circular orbit at an altitude of 6.46×10^5 meters (400 miles). To prepare for docking, the spacecraft must decrease its altitude 16,150 meters (10 miles) and remain in a circular orbit. How much work in joules must retrorockets do on the spacecraft to make this maneuver?

2. (a) What limit must $\frac{R_1v^2}{2GM}$ approach in the equation

   $$R_2 = \frac{R_1}{1 - \dfrac{R_1v^2}{2GM}}$$

   if R_2 is to become infinitely large?

   (b) Set $\frac{R_1v^2}{2GM}$ equal to this limit and compute the value of v required for R_2 to become infinitely large.

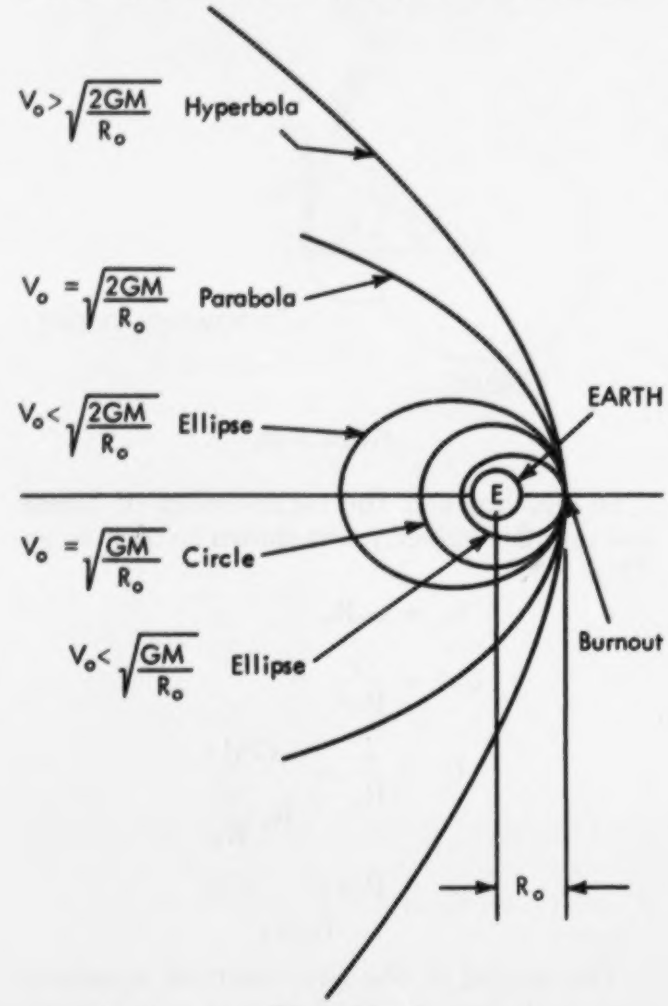
3. Compute the escape velocity for a space-vehicle at (a) an altitude of 4.845×10^5 meters (300 miles) and (b) at an altitude of 1.292×10^6 meters (800 miles).

4. A satellite is in a circular orbit with a period of 110 minutes. How much additional velocity must the satellite be given to escape from Earth?

5–7 Satellite Paths

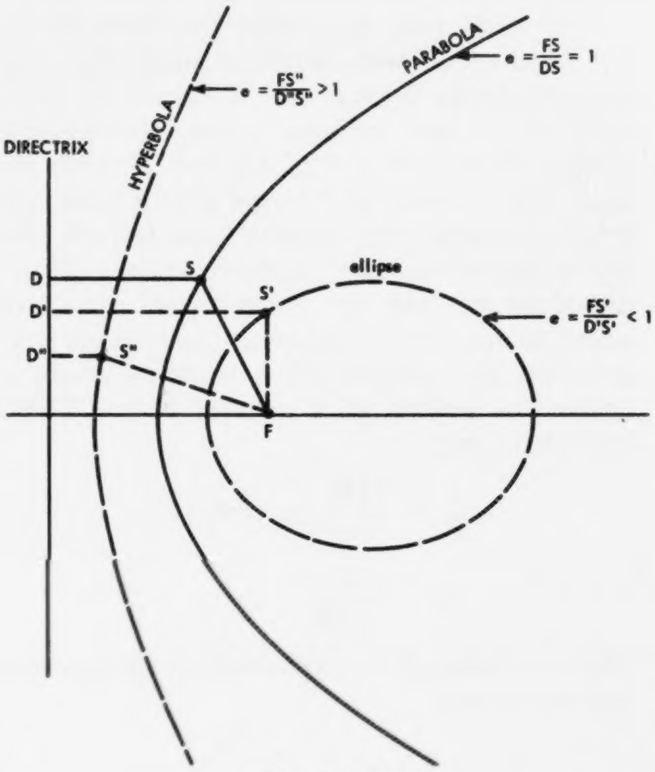
In Sections 5–4 and 5–5 we considered primarily the mechanics of circular orbits. We will now consider more general orbits of satellites; that is, orbits that are not circular. It will be assumed (1) that the satellite moves under the action of a central force, (2) that this force acts from a fixed point, and (3) that the motion of the satellite is in one plane. The paths that a satellite may assume at burnout are illustrated in Figure 5–17. The path that is assumed depends upon the satellite's velocity at burnout. As a first step toward understanding the motion of a satellite along these paths, we will review the geometry of the paths that is pertinent to our problem.

The path of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed line is called a *conic section* (Section 1–8), or simply a *conic*. The fixed point F (Figure 5–18) is called the *focus* of the conic, the fixed line the *directrix*, and the constant ratio $\overline{FS}/\overline{DS}$ is the *eccentricity* of the conic.



$V_o > \sqrt{\frac{2GM}{R_o}}$ Hyperbola
$V_o = \sqrt{\frac{2GM}{R_o}}$ Parabola
$V_o < \sqrt{\frac{2GM}{R_o}}$ Ellipse
EARTH
E
$V_o = \sqrt{\frac{GM}{R_o}}$ Circle
$V_o < \sqrt{\frac{GM}{R_o}}$ Ellipse
Burnout
R_o

Figure 5–17



$e = \frac{FS''}{D''S''} > 1$
PARABOLA
$e = \frac{FS}{DS} = 1$
DIRECTRIX
HYPERBOLA
ellipse
D
D'
D''
S
S'
S''
$e = \frac{FS'}{D'S'} < 1$
F

Figure 5–18

The conic sections fall into four classes according to their eccentricities. These classes are for the parabola, hyperbola, ellipse, and circle. Broken lines have been used to draw the ellipse and hyperbola in Figure 5–18. The focus F and the directrix DD′ is common to all three conics. The eccentricity e for each class may be identified as follows:

parabola, $e = 1$
hyperbola, $e > 1$
ellipse, $e < 1$
circle, $e = 0$

The definitions of the ellipse given here and in Section 2–4 are equivalent alternative definitions.

The motion of a satellite in a circular path is described by a velocity whose magnitude is constant but whose direction is continually changing. The satellite experiences a steady acceleration in the direction of a central force. If a satellite is moving along the path of a conic that is not a circle, both the magnitude and direction of its velocity changes continually. The satellite is accelerated by a central force whose magnitude and direction changes continually. It is almost needless to say that the motion of a satellite along the path of a conic is very complex, and it is only by the use of higher mathematics that an equation of the motion can be derived.

For years you have used the formula for the area of a circle without understanding completely its derivation. We shall use Einstein's $E = mc^2$ without having the faintest notion as to how it was derived. Should we close this chapter and forget about elliptical orbits because we cannot understand the initial equations used to derive algebraic expressions we can all understand? You are asked to accept two important equations that describe the motion of a satellite moving under the action of a central force. These equations are:

$$\frac{1}{R} = \frac{GM}{R^4\omega^2} + C\cos\theta$$

$$e = \frac{CR^4\omega^2}{GM}$$

The symbols will be explained as we consider the equations.

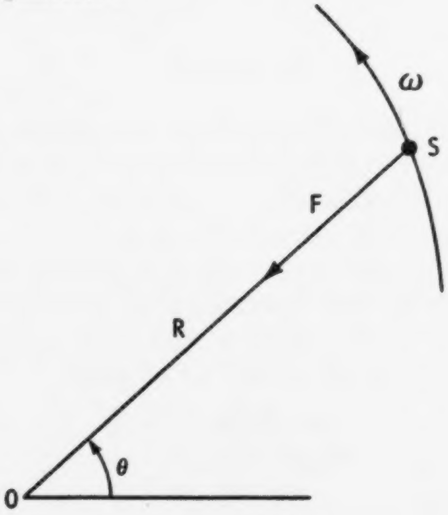


Figure 5–19

Figure 5–19 illustrates a satellite of mass m moving under the action of a central force F. This force is directed toward the point 0 and is the gravitational force $G\frac{mM}{R^2}$. The position of the satellite is described by polar coordinates (Section 2–3) θ and R, the magnitude of the radius vector OS. The angular velocity (Section 4–8) of the satellite at that instant is ω.

The two equations just given to you describe the motion of the satellite. The letter C in these equations is a constant obtained from the solution of the initial equations setup to describe the satellite's motion. The value of C must be found before either equation can be fully interpreted. This is done by setting $\theta = 0°$ and launching the satellite parallel to Earth as illustrated by Figure 5–20. The distance from the center of Earth to the satellite when $\theta = 0°$ is R_0; the linear velocity of the satellite is v_0; and the angular velocity of the satellite is ω_0. Since $\cos 0° = 1$, the first equation becomes

$$\frac{1}{R_0} = \frac{GM}{R_0^4\omega_0^2} + C$$

$$C = \frac{1}{R_0} - \frac{GM}{R_0^4\omega_0^2}$$

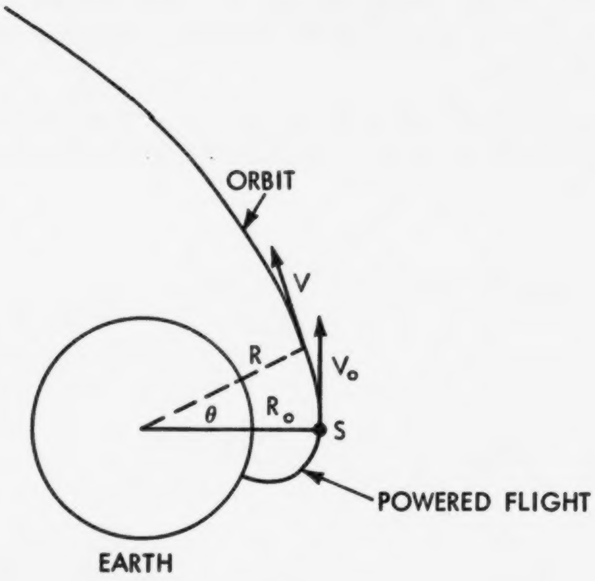


Figure 5–20

In Section 4–8 the relationship of linear and angular velocity was shown to be $v = \omega r$. Thus

$$v_0 = \omega_0 R_0$$

$$\omega_0^2 = \frac{v_0^2}{R_0^2}$$

$$C = \frac{1}{R_0} - \frac{GM}{R_0^4\frac{v_0^2}{R_0^2}}$$

$$C = \frac{R_0v_0^2 - GM}{R_0^2v_0^2}$$

The second of the two assumed equations reads as follows for the initial ($\theta = 0°$) conditions of flight:

$$e = \frac{CR_0^4\omega_0^2}{GM}$$

After substituting for C and ω_0^2 we get for the eccentricity

$$e = \frac{\left(\dfrac{R_0v_0^2 - GM}{R_0^2v_0^2}\right) R_0^4 \dfrac{v_0^2}{R_0^2}}{GM}$$

$$e = \frac{R_0v_0^2 - GM}{GM}$$

The class of the orbit can be determined by substituting values for v_0, the velocity of the satellite at burnout. If the escape velocity $\sqrt{\frac{2GM}{R_0}}$ is substituted for v_0,

$$e = \frac{R_0\dfrac{2GM}{R_0} - GM}{GM}$$

$$e = \frac{2GM - GM}{GM} = 1$$

The eccentricity obtained is the eccentricity for a parabola! The paths listed below can also be predicted by substituting the velocity with which it is identified.

If $v_0 > \sqrt{\frac{2GM}{R_0}}$, $e > 1$ path is hyperbolic.

If $v_0 < \sqrt{\frac{2GM}{R_0}}$, $e < 1$ path is elliptical.

If $v_0 = \sqrt{\frac{GM}{R_0}}$, $e = 0$ path is circular.

Figure 5–17 shows graphically the result for each case. When the burnout velocity v_0 is less than $\sqrt{\frac{2GM}{R_0}}$ but greater than $\sqrt{\frac{GM}{R_0}}$, the satellite will go into an elliptical orbit at perigee (Figure 5–17). If the burnout velocity is less than $\sqrt{\frac{GM}{R_0}}$, the satellite will still go into an elliptical orbit but the point of burnout will be at apogee (Figure 5–17). This becomes apparent when values smaller than $\sqrt{\frac{GM}{R_0}}$ are substituted for v_0 in the eccentricity equation. The result is a negative value for e, an indication that the center of force has shifted to the other focus of the ellipse. If the burnout velocity falls too far below $\sqrt{\frac{GM}{R_0}}$, the elliptical path will intersect Earth and the satellite will not go into orbit.

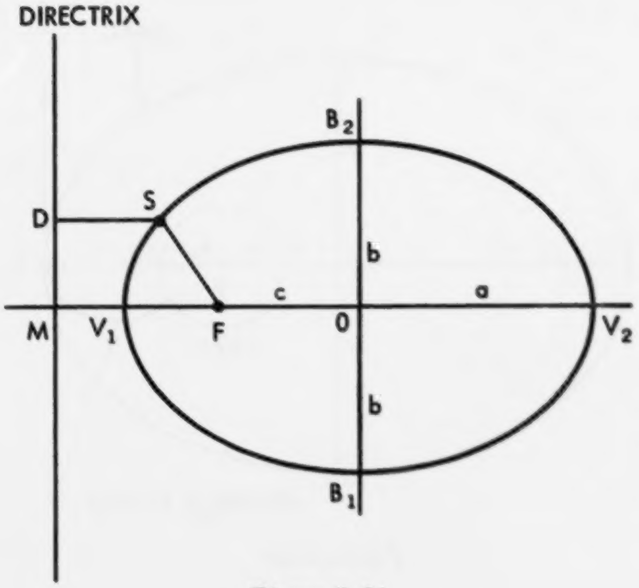


Figure 5–21

It is convenient to express the motion and energy equations for a body moving in an elliptical path in terms of a, the semimajor axis rather than in terms of e, the eccentricity. Figure 5–21 will be used to determine the relationship of e and a. Line segment $\overline{V_1V_2}$ is the major axis of the ellipse and has length 2a. Line segment $\overline{B_1B_2}$ is the minor axis and has length 2b. These two axes intersect at 0, the center of the ellipse. The distance from focus F to the center of the ellipse is c. By definition

$$e = \frac{\overline{FS}}{\overline{DS}}$$

$$\overline{FS} = e\,\overline{DS}$$

for each point S of the ellipse. In particular at V_1, and V_2

$$\overline{OV_1} - \overline{OF} = e(\overline{OM} - \overline{OV_1})$$

$$\overline{V_2O} + \overline{OF} = e(\overline{V_2O} + \overline{OM})$$

where $a = \overline{V_2O} = \overline{OV_1}$

$c = \overline{OF}$

$d = \overline{OM}$

Rewriting the equations with a, c, and d results in

$$a + c = e(d + a)$$

$$a - c = e(d - a)$$

After removing parentheses and subtracting the second equation from the first we obtain

$$c = a\,e$$

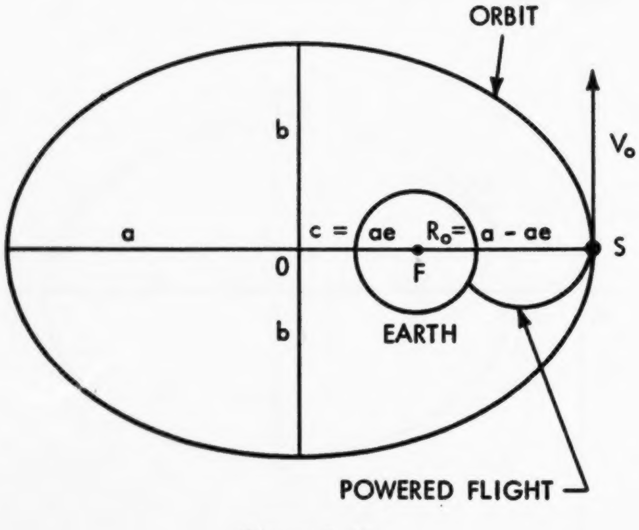


Figure 5-22

Figure 5-22 illustrates a spacecraft S going into an elliptical orbit. The distance from the center of the orbit to the center of Earth is

$$\overline{OF} = c = a\,e$$

The distance from the center of Earth to the spacecraft is

$$R_0 = a - a\,e$$

$$e = \frac{a - R_0}{a}$$

After substituting this value for e in the eccentricity equation we have

$$\frac{a - R_0}{a} = \frac{R_0 v_0^2 - GM}{GM}$$

$$aGM - R_0GM = aR_0v_0^2 - aGM$$

$$v_0^2 = \frac{2GM}{R_0} - \frac{GM}{a}$$

$$v_0^2 = GM\left(\frac{2}{R_0} - \frac{1}{a}\right),$$

the equation for the motion of the spacecraft in its elliptical orbit. Remember that v_0 and R_0 are simply the instantaneous magnitudes of v and R when the satellite is at perigee. The equation

$$v^2 = GM\left(\frac{2}{R} - \frac{1}{a}\right)$$

expresses the velocity v of the spacecraft at any distance R in meters from the center of Earth, where v is the velocity of the satellite in m/sec, M is the mass of Earth in kilograms, G is the gravitational constant in $\frac{\text{newton (m)}}{\text{kg}^2}$, and a is the semimajor axis of the orbit in meters.

The time-rate of change in the area swept out by the radius vector as the satellite moves through its orbit is called the *areal velocity*. The ratio of the area of the ellipse to the areal velocity defines the period of a satellite in an elliptical orbit. The period of an Earth satellite in an elliptical orbit is computed using the formula

$$T^2 = \frac{4\pi^2a^3}{GM}$$

where T is the period in seconds, M is the mass of Earth in kilograms, a is the semimajor axis of the orbit in meters, and G is the gravitational constant in $\frac{\text{newton (m)}}{\text{kg}^2}$.

The total energy of a satellite in an elliptical orbit is the sum of its kinetic and gravitational potential energies. The kinetic energy may be found as follows:

$$v^2 = GM\left(\frac{2}{R} - \frac{1}{a}\right)$$

$$(1/2)m\,v^2 = (1/2)m\,GM\left(\frac{2}{R} - \frac{1}{a}\right)$$

$$(1/2)m\,v^2 = GmM\left(\frac{1}{R} - \frac{1}{2a}\right)$$

$$E_k = GmM\left(\frac{1}{R} - \frac{1}{2a}\right)$$

The gravitational potential energy is

$$E_p = GmM\left(\frac{1}{r} - \frac{1}{R}\right)$$

The total energy is therefore

$$E = GmM\left(\frac{1}{R} - \frac{1}{2a}\right) + GmM\left(\frac{1}{r} - \frac{1}{R}\right)$$

$$E = \frac{GmM}{R} - \frac{GmM}{2a} + \frac{GmM}{r} - \frac{GmM}{R}$$

$$E = GmM\left(\frac{1}{r} - \frac{1}{2a}\right) \tag{1}$$

where E is the total energy in joules, G is the gravitational constant in $\frac{\text{newton (m)}}{\text{kg}^2}$, m is the mass of the satellite in kilograms, M is the mass of Earth in kilograms, r is the radius of Earth in meters, and a is the semimajor axis of the orbit in meters. The energy equation (1) shows that the total energy of a satellite in an elliptical orbit is a function of only the semimajor axis a and is independent of the shape of the orbit: This is illustrated graphically in Figure 5-23. The three orbits have the same semimajor axis a. Hence satellites in these three orbits will have the same energy.

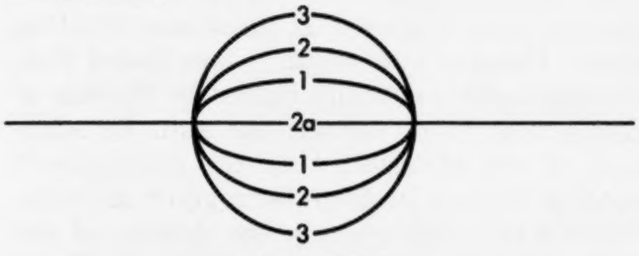


Figure 5-23

The velocity equation

$$v^2 = GM\left(\frac{2}{R} - \frac{1}{a}\right)$$

shows that a satellite in an elliptical orbit has the greatest velocity when R is least; that is, when the satellite is at perigee. At apogee the distance R is maximum so the satellite's velocity is minimum. The mathematical relationship of these two velocities is interesting. Figure 5-22 shows that distance R for perigee is a − ae. For apogee, the distance R is

$$ae + ae + (a - ae)$$

or simply ae + a. When these distances are substituted in the velocity equation you obtain

$$v_p^2 = GM\left(\frac{2}{a - ae} - \frac{1}{a}\right)$$

$$= GM\left[\frac{2a - (a - ae)}{(a - ae)a}\right]$$

$$v_a^2 = GM\left(\frac{2}{ae + a} - \frac{1}{a}\right)$$

$$= GM\left[\frac{2a - (ae + a)}{(ae + a)a}\right]$$

$$\frac{v_p^2}{v_a^2} = \frac{GM\left[\frac{2a - (a - ae)}{(a - ae)a}\right]}{GM\left[\frac{2a - (ae + a)}{(ae + a)a}\right]}$$

$$\frac{v_p^2}{v_a^2} = \frac{2a - a + ae}{(a - ae)a} \times \frac{(ae + a)a}{2a - ae - a}$$

$$\frac{v_p^2}{v_a^2} = \frac{a + ae}{a - ae} \times \frac{a + ae}{a - ae}$$

$$\frac{v_p^2}{v_a^2} = \frac{(a + ae)^2}{(a - ae)^2}$$

$$\frac{v_p}{v_a} = \frac{a + ae}{a - ae} = \frac{a\,(1 + e)}{a\,(1 - e)}$$

$$\frac{v_p}{v_a} = \frac{1 + e}{1 - e}$$

where v_p is the velocity at perigee, v_a is the velocity at apogee, and e is the eccentricity of the orbit.

5-7 *Exercises Satellite Paths*

Use these values in the exercises:

$$G = 6.670 \times 10^{-11}\,\frac{\text{newton (m)}}{\text{kg}^2}$$

Radius of Earth = 6.37×10^6 meters

Mass of Earth = 6×10^{24} kilograms

1. Show that the eccentricity of an elliptical orbit may be expressed as

   $$e = \frac{v_p - v_a}{v_p + v_a}$$

   where v_p is the velocity of the satellite at perigee and v_a is the velocity at apogee.
2. Find the sum of the kinetic and gravitational potential energies of a satellite in a circular orbit of radius *a*. How does this energy compare with the energy the satellite would have if it was in an elliptical orbit of semimajor axis *a*?
3. In Chapter 4 linear momentum was defined as the product of mass and linear velocity. This was expressed as $\overrightarrow{mv}$. The momentum of a body moving in a curved path is called moment of momentum, or angular momentum. Its magnitude is the product of the magnitude of the linear momentum and the perpendicular distance between the line of motion of the body and the center about which the body moves. This is illustrated by Figure 5-24.

   A satellite S of mass m has a tangential (linear) velocity $\vec{v}$ when it is at a distance R from 0, the center of force. The

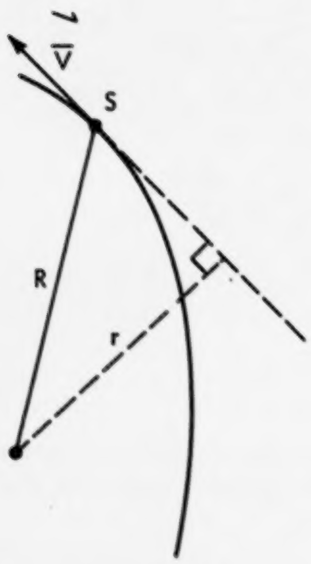


Figure 5-24

magnitude of its linear momentum at that instant is mv and the magnitude of its angular momentum is mvr.

When an Earth satellite is at apogee and perigee, the distance r in the expression mvr is the distance from the satellite to the center of Earth. This distance may be expressed in terms of a and e. Write expressions for the angular momenta at apogee and perigee. Equate these expressions and reduce to lowest terms. What do you observe? What conclusion might you draw?

4. On January 25, 1964 NASA launched Echo II, an orbit-inflatable plastic balloon to test its ability to reflect radio signals. At perigee this satellite is 1.037×10^6 meters (642 miles) from the surface of Earth while at apogee its altitude is 1.318×10^6 meters (816 miles). Compute (a) the velocity of this satellite at apogee and perigee, (b) the eccentricity of its orbit, and (c) the period of this satellite.

5–8 Orbit determination

Computing satellite orbits by theoretical methods is only the beginning of orbit determination. The analysis of data obtained from satellite tracking stations has revealed that there are a number of environmental factors which cause perturbations (changes) in orbits. Some of these factors are:

(1) atmospheric drag
(2) atmospheric bulge
(3) variable atmosphere
(4) Earth's pear-shape
(5) radiation pressure
(6) the gravitational force due to the moon and sun.

The drag effect of the atmosphere occurs primarily in the neighborhood of perigee. This causes the satellite to lose energy which, in turn, results in a decrease in the orbital period and the apogee height. The formula for the period of a satellite in an elliptical orbit shows that the semi-major axis of the orbit will decrease if the period of the satellite decreases. Figure 5–25 illustrates the drag effect. Using orbital data, it was found that the atmosphere actually bulges on the side of Earth that faces toward the sun. In addition, it was observed that the atmospheric density was not uniform for a given altitude. There was a difference in the density of the air on the dark and light sides of Earth. There was even some variation with longitude and latitude. It was also observed that solar activity caused significant changes in atmospheric density.

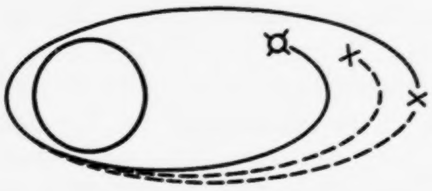


Figure 5-25

The pear-shape of Earth causes small deviations in the orbit of a satellite. The "out-of-roundness" results in increased accelerations and decelerations as the satellite approaches and recedes from the bulge respectively.

The pressure of radiation on a satellite is small and may seem negligible but acting over a long period, however, it does produce a detectable perturbation, even on satellites of ordinary mass. The effect on a low-mass satellite such as an Echo satellite is quite significant. To understand the pressure of radiation one must look to the nature of radiation. In many experiments, radiation behaves as though it is propagated as a wave, so we describe radiation in terms of wave velocity, frequency, and wavelength. These three properties have the relationship

$$C = f\lambda$$

where C is the velocity of the radiation in free space, f is the frequency of the wave, and

λ is the wavelength. In other experiments, radiation behaves as though it is propagated through space as discrete bundles of energy called photons. A photon of energy is represented by

$$E = hf$$

where E is the energy of a photon in ergs; h is Planck's constant; that is, 6.623×10^{-27} ergs (sec); and f is the frequency of the radiation. This dual nature of radiation is one of the mysteries of science. A photon has no mass at rest but it does have a mass when moving and momentum. The momentum of a photon is $\frac{h}{\lambda}$ where h is Planck's constant and λ is the wavelength of the radiation.

It would seem that the moon and sun are too distant to cause perturbations of a small satellite close to Earth. Such is not the case, however.

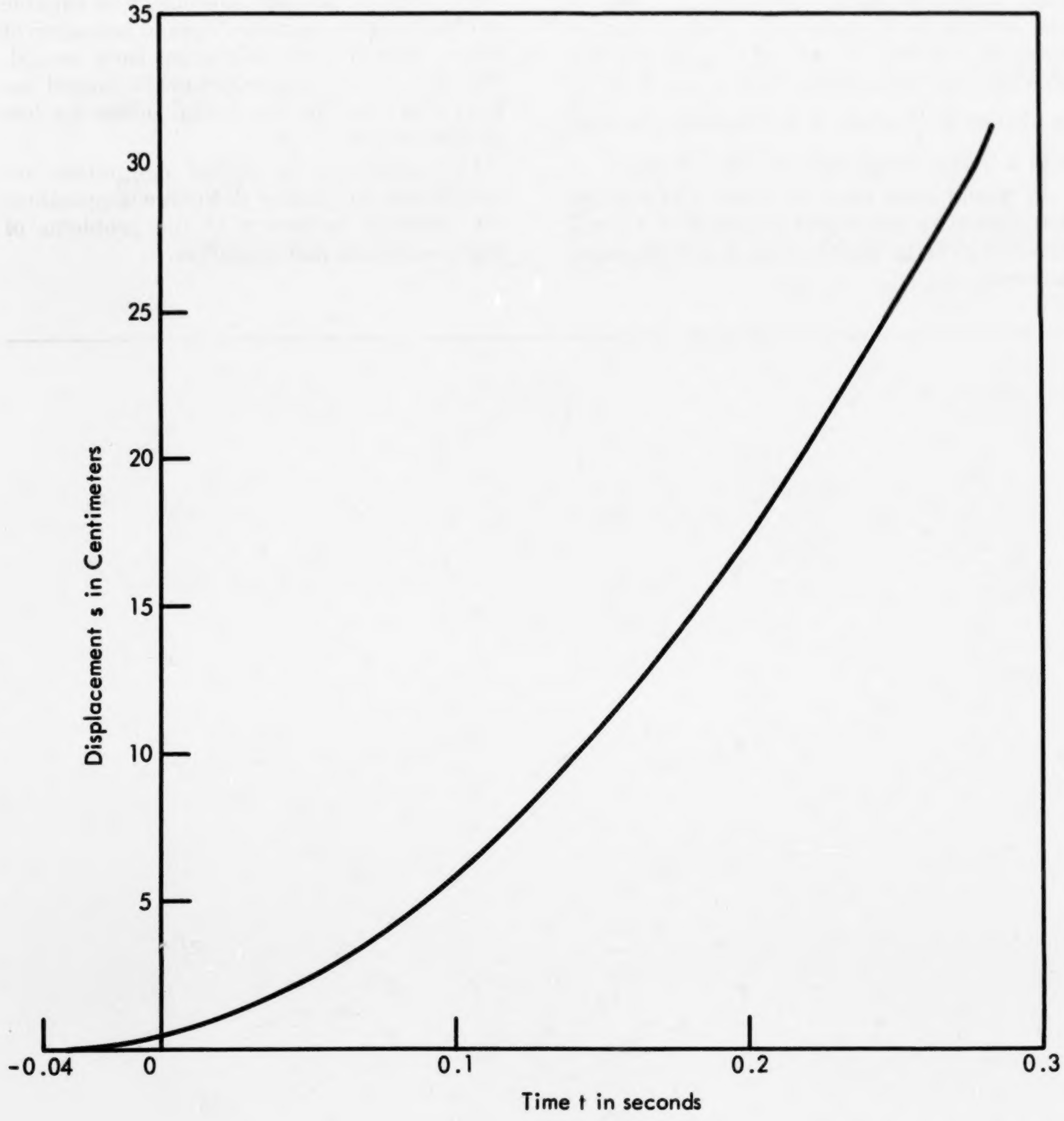
Because of the perturbations in a satellite's orbit, radar tracking systems are employed to gather orbital data. This data is fed into high speed computers and the trajectory of the orbit is computed. Orbit determination is so critical for manned spaceflights that each orbit trajectory must be determined within seconds of the time when the spacecraft is actually in orbit. NASA's radar tracking system for Project Mercury was capable of obtaining ten complete sets of measures of range, azimuth, and elevation each second. The data for the computed orbits lagged behind the data for the actual orbits by less than a minute.

Computations by digital computers are considered in Chapter 6. Such computations are essential for many of the problems of space scientists and engineers.

DETAILED SOLUTIONS FOR QUESTIONS ON SPACE MECHANICS

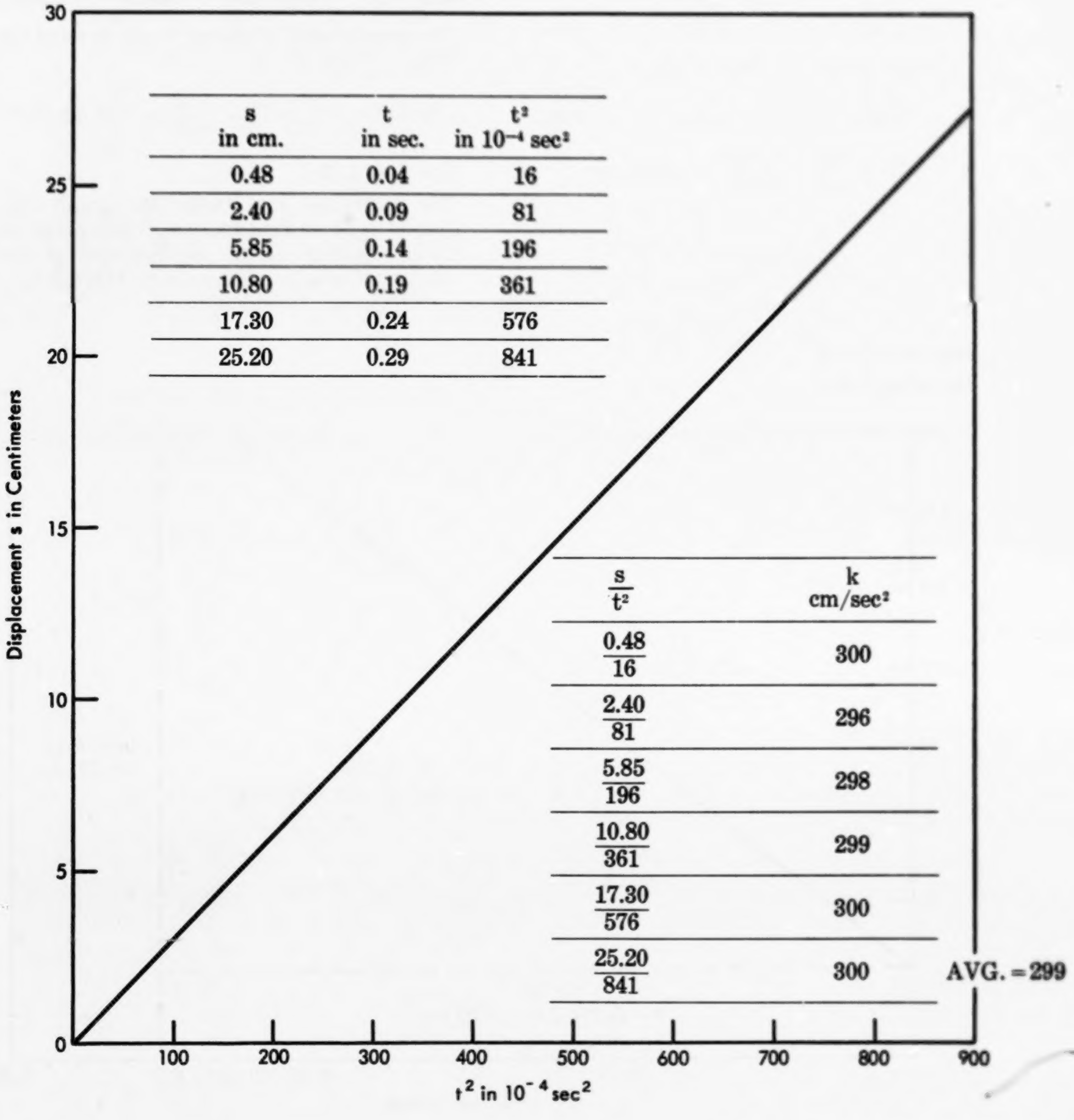
Question 5–1–1

The time it took the car to travel from the rest point to the first impact mark thereafter is extrapolated by projecting the displacement-time graph to zero. This time is approximately 0.04 seconds.



Question 5–1–2

The relationship of s and t must be determined by trial and error. Since the graph of question 5–1–1 appears to be a parabola, the first choice would be to graph S as a function of t^2. Time t should now include the time extrapolated from the graph of question 5–1–1. The graph obtained shows that $s \propto t^2$. Hence $\frac{s}{t^2} = k$, or $s = kt^2$. The constant k is determined by dividing s by t^2. The constant k turns out to be about 299, so equation $s = 299t^2$ essentially fits the graph.



s in cm.	t in sec.	t^2 in 10^{-4} sec^2
0.48	0.04	16
2.40	0.09	81
5.85	0.14	196
10.80	0.19	361
17.30	0.24	576
25.20	0.29	841

$\frac{s}{t^2}$	k cm/sec^2
$\frac{0.48}{16}$	300
$\frac{2.40}{81}$	296
$\frac{5.85}{196}$	298
$\frac{10.80}{361}$	299
$\frac{17.30}{576}$	300
$\frac{25.20}{841}$	300

AVG. = 299

Question 5–1–3

The approximate instantaneous velocity of the car at 0.14 seconds from rest may be found using the formula $s = 299t^2$. The instantaneous velocity of the car is essentially equal to its average velocity for a very short period of time where 0.14 seconds is the initial time for the period.

Initial Value of t	New Value of t	Δt
0.140	0.141	0.001

Initial Value of s	New Value of s	Δs
5.860	5.944	0.084

$$V = \frac{\Delta s}{\Delta t} = \frac{0.084}{0.001} = 84 \text{ cm/sec}$$

Question 5–1–4

Time Interval	Δt	Δs	$V = \frac{\Delta s}{\Delta t}$	Time at Middle of Time Interval
0–0.05	0.05	1.92	38.4	0.025
0.05–0.10	0.05	3.45	69.0	0.075
0.10–0.15	0.05	4.95	99.0	0.125
0.15–0.20	0.05	6.50	130.0	0.175
0.20–0.25	0.05	7.90	158.0	0.225

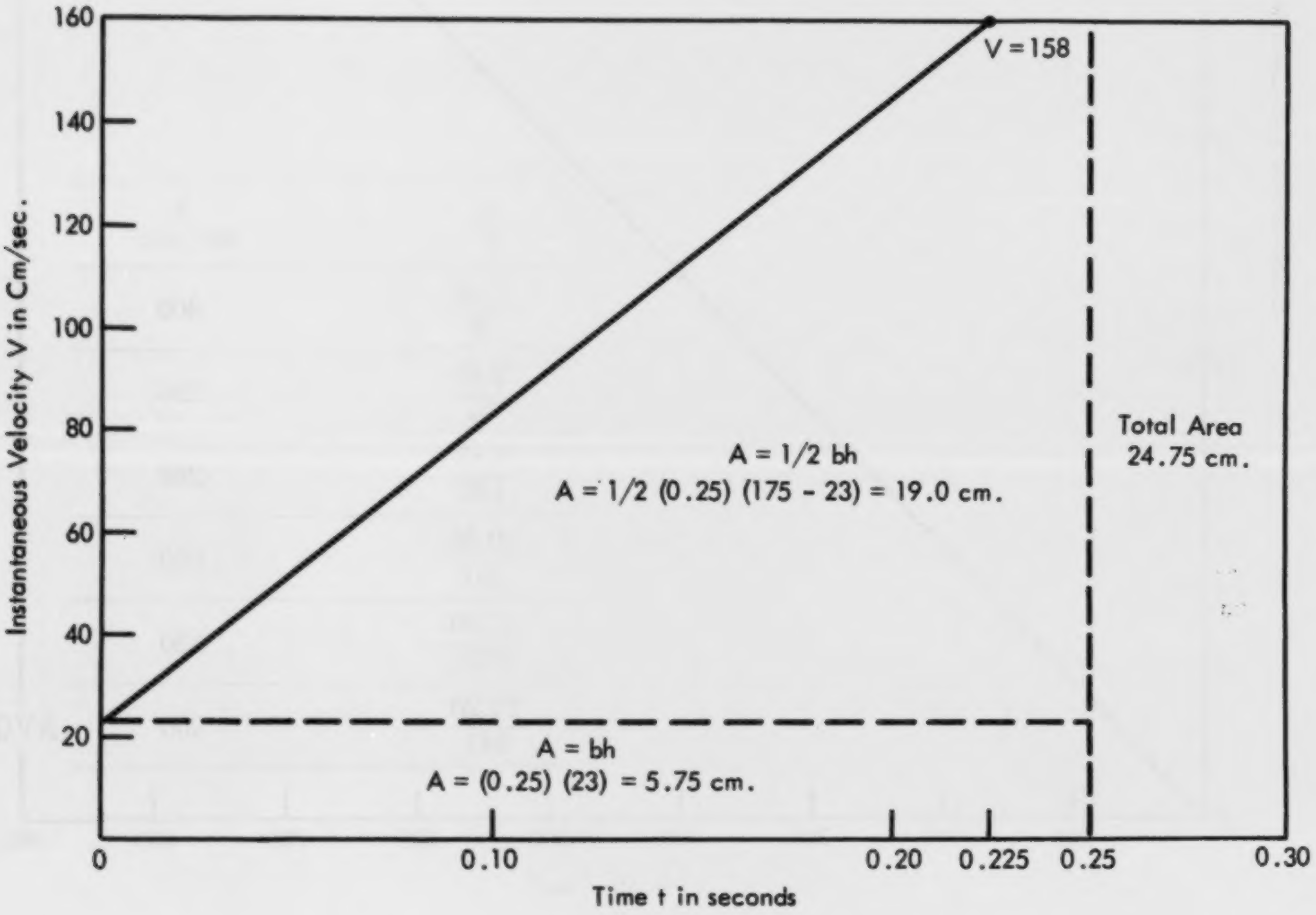
The acceleration of the car is the slope of the velocity-time graph.

$$\text{Slope} = \frac{175 - 23}{0.25} = \frac{152}{0.25} = 608 \text{ cm/sec}^2$$

Question 5–2–1

The estimated area under the graph (see graph) is 24.75 centimeters as compared to 24.72 centimeters, the displacement of the car as indicated by the data of Table 5–1.

Question 5–1–4
Question 5–2–1



Question 5–2–2

(a) $V_f = V_o + at$

$22 = 88 - 10t$

$10t = 66$

$t = 6.6$ seconds

(b) $s = V_ot + \frac{1}{2}at^2$

$s = 88(6.6) - \frac{1}{2}(10)(6.6)(6.6)$

$s = 580.8 - 217.8$

$s = 363$ feet

Question 5–2–3

(a) $s = V_ot + \frac{1}{2}at^2$

$V_o = 0$ if body falls freely from rest

$s = \frac{1}{2}at^2$

$s = \frac{1}{2}(32)(1^2) = 16$ feet

$s = \frac{1}{2}(32)(3^2) = 144$ feet

$s = \frac{1}{2}(32)(5^2) = 400$ feet

(b) $V_f = V_o + at$

$V_o = 0$

$V_f = at$

$V_f = 32(1) = 32$ ft/sec.

$V_f = 32(3) = 96$ ft/sec.

$V_f = 32(5) = 160$ ft/sec.

Question 5–2–4

(a) $V_f = V_o + at$

$V_f = 0$ at the peak

$0 = 96 - 32t$

$32t = 96$

$t = 3$ seconds

(b) $s = V_ot + \frac{1}{2}at^2$

$s = 96(3) - \frac{1}{2}(32)(3^2)$

$s = 288 - 144$

$s = 144$ feet

(c) $s = V_ot + \frac{1}{2}at^2$

$80 = 96t - \frac{1}{2}(32)t^2$

$80 = 96t - 16t^2$

$16t^2 - 96t + 80 = 0$

$t^2 - 6t + 5 = 0$

$(t - 5)(t - 1) = 0$

$t - 1 = 0$

$t = 1$ second after projection

$t - 5 = 0$

$t = 5$ seconds after projection

Question 5–2–5

$V_f = V_o + at$

square both sides of equation

$V_f^2 = V_o^2 + 2V_oat + a^2t^2$

$s = V_ot + \frac{1}{2}at^2$

multiply by 2a

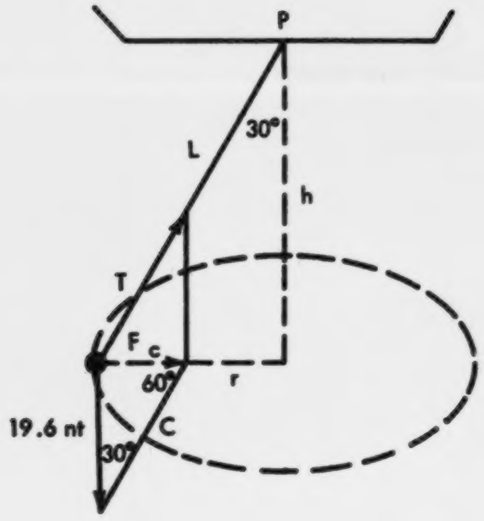
$2as = 2V_oat + a^2t^2$

Subtract the second equation from the first

$$\begin{array}{rl} V_f^2 = & V_o^2 + 2V_oat + a^2t^2 \\ 2as = & \phantom{V_o^2 +{}} 2V_oat + a^2t^2 \\ \hline V_f^2 - 2as = & V_o^2 \\ V_f^2 = & V_o^2 + 2as \end{array}$$

Question 5–3–1

(a) and (b)



(c) In the case of a 30°, 60° right triangle; the side opposite the 60° angle equals one-half the hypotenuse C times $\sqrt{3}$.

$$\tfrac{1}{2}C\sqrt{3} = 19.6$$
$$C\sqrt{3} = 39.2$$
$$C = \frac{39.2}{\sqrt{3}} = \frac{39.2\sqrt{3}}{3}$$
$$C = 22.63 \text{ newtons}$$

The side opposite the 30° angle equals one-half the hypotenuse

$$F_c = \tfrac{1}{2}C$$
$$F_c = \tfrac{1}{2}(22.63)$$
$$F_c = 11.32 \text{ newtons}$$

An alternate method

$$\tan 30° = \frac{F_c}{19.6}$$
$$F_c = 19.6 \tan 30°$$
$$F_c = 19.6(0.57735)$$
$$F_c = 11.32 \text{ newtons}$$

(d) The triangle formed by L, h, and r is a 30°, 60° right triangle. The circle has a radius r equal to one-half the hypotenuse L.

$$L = 1 \text{ meter}$$
$$r = \tfrac{1}{2}L = \tfrac{1}{2} \text{ meter}$$
$$F_c = \frac{mV^2}{r}$$
$$V^2 = \frac{F_c r}{m}$$
$$V = \sqrt{\frac{F_c r}{m}} = \sqrt{\frac{11.32(0.5)}{2}} = \sqrt{2.83}$$
$$V \approx 1.68 \text{ m/sec.}$$

Question 5–3–2

The centripetal force needed to keep the 1 kilogram mass in the circle while overhead ordinarily

the weight force of the mass + the down force of the bucket against the mass

At the least overhead velocity

the weight force of the mass = centripetal force

$$mg = F_c = \frac{mV^2}{r}$$
$$V = \sqrt{gr} = \sqrt{9.8(0.8)}$$
$$V = \sqrt{7.84}$$
$$V \approx 2.8 \text{ m/sec.}$$

Question 5-4-1

$$\frac{F_1}{F_2} = \frac{R_2^2}{R_1^2}$$

$$\frac{W}{\frac{3}{4}W} = \frac{R_2^2}{(4000)^2}$$

$$\frac{4}{3} = \frac{R_2^2}{(4000)^2}$$

$$R_2^2 = \frac{4(4000)^2}{3} = \frac{64 \times 10^6}{3}$$

$$R_2 = \frac{8 \times 10^3}{\sqrt{3}} = \frac{8 \times 10^3\sqrt{3}}{3} \approx 4.618 \times 10^3 \text{ miles}$$

$$R_2 \approx 4618 \text{ miles}$$

At an altitude of 618 miles your weight is $\frac{3}{4}$ of what it is at the surface of the earth.

Question 5-4-2

$$\frac{g_1}{g_2} = \frac{R_2^2}{R_1^2}$$

$$\frac{32}{g_2} = \frac{(8000)^2}{(4000)^2}$$

$$g_2 = \frac{32(4000)(4000)}{(8000)(8000)}$$

$$g_2 = \frac{32}{4} = 8 \text{ ft/sec}^2$$

Question 5-4-3

$$F = G\frac{mM}{r^2}$$

The gravitational force F on a small mass m at the surface of the earth equals mg.

$$mg = G\frac{mM}{r^2}$$

$$M = \frac{gr^2}{G}$$

$$M = \frac{(9.8)(6.37 \times 10^6)^2}{6.670 \times 10^{-11}} = \frac{3.98 \times 10^{14}}{6.670 \times 10^{-11}}$$

$$M \approx 5.97 \times 10^{24} \text{ kilograms } (6.57 \times 10^{21} \text{ tons})$$

Question 5-4-4

$$M = 80m$$

$$R = 240{,}000 - r$$

$$F = G\frac{m'M}{R^2} \qquad F = G\frac{m'm}{r^2}$$

$$G\frac{m'M}{R^2} = G\frac{m'm}{r^2}$$

$$Gm'Mr^2 = Gm'mR^2$$

$$Mr^2 = mR^2$$

$$M = 80m$$

$$R^2 = (240{,}000 - r)^2$$

$$80mr^2 = m(240{,}000 - r)^2$$

$$79r^2 + 480{,}000r - (240{,}000)^2 = 0$$

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{(-4.8 \times 10^5) \pm \sqrt{(4.8 \times 10^5)^2 + (4)(79)(2.4 \times 10^5)^2}}{2(79)}$$

$$r = \frac{(-4.8 \times 10^5) \pm \sqrt{1843 \times 10^{10}}}{158}$$

$$r \approx \frac{(-4.8 \times 10^5 \pm (42.9 \times 10^5)}{158}$$

$$r \approx 24{,}100 \text{ miles}$$

$$R \approx 215{,}900 \text{ miles}$$

Question 5-4-5

The centripetal force required equals the gravitational force, or the weight force mg.

$$F_c = \frac{mV^2}{R} = G\frac{mM}{R^2} = F = mg$$

$$mg = \frac{mV^2}{R}$$

$$V^2 = gR$$

$$V = \sqrt{gR}$$

Question 5-4-6

(a) $V = \sqrt{\frac{GM}{R}}$

$$V = \sqrt{\frac{(6.670 \times 10^{-11})(6 \times 10^{24})}{6.853 \times 10^6}}$$

$$V \approx \sqrt{\frac{40.02 \times 10^{13}}{6.853 \times 10^6}} = \sqrt{58.4 \times 10^6}$$

$V \approx 7{,}640$ m/sec. (25,059 ft/sec or 17,083 mi/hr)

(b) $V = \sqrt{\frac{GM}{R}}$

$$V \approx \sqrt{\frac{(6.670 \times 10^{-11})(6 \times 10^{24})}{80(2.223 \times 10^6)}}$$

$$V \approx \sqrt{\frac{40.02 \times 10^{13}}{177.8 \times 10^6}} \approx \sqrt{2.25 \times 10^6}$$

$V \approx 1{,}500$ m/sec. (4,920 ft/sec or 3,354 mi/hr)

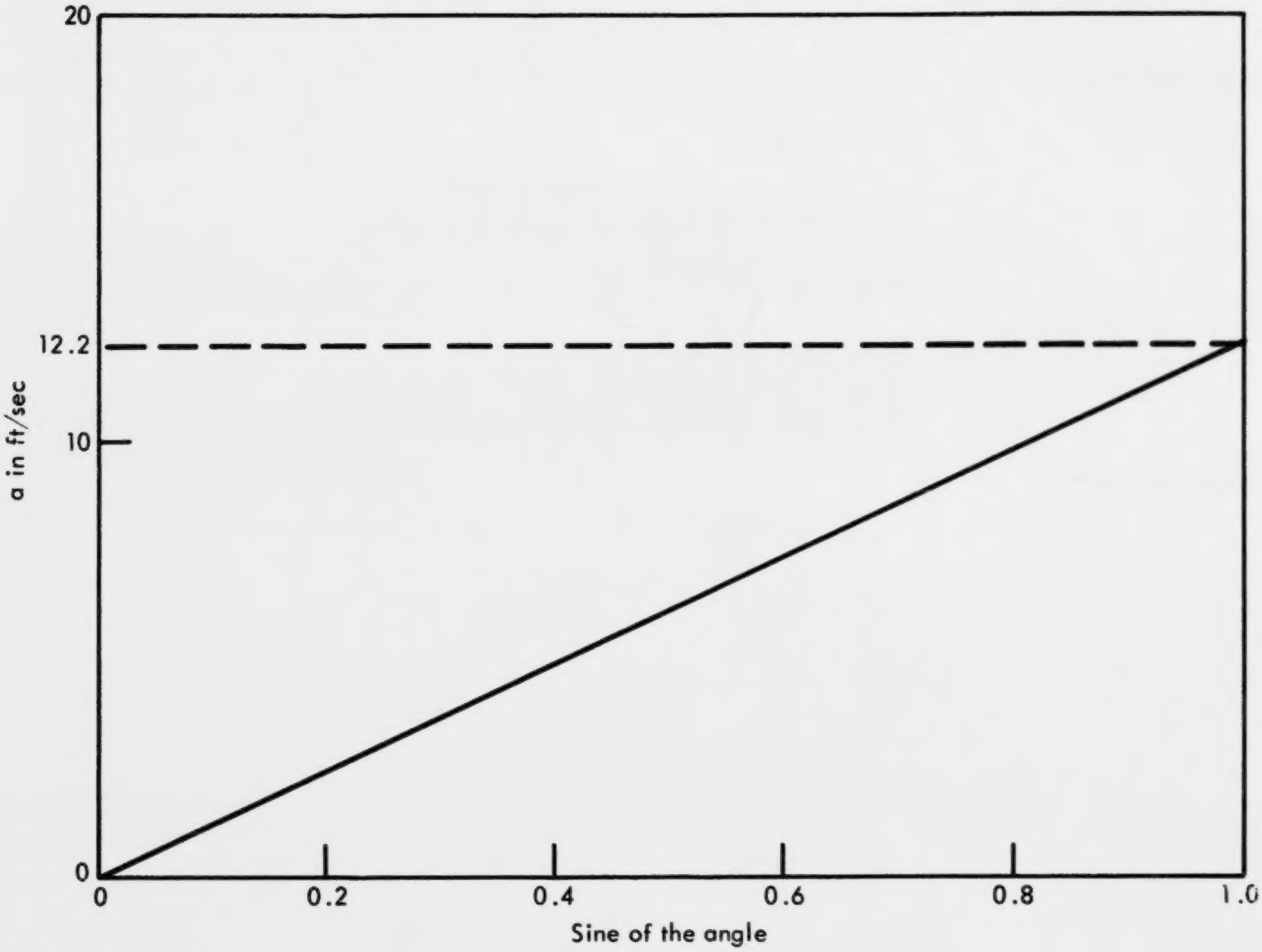
Question 5-4-7

(a)

a in ft/sec²	Angle	Sine of angle
3.15	15°	0.2588
6.08	30°	0.5000
9.31	50°	0.7660
11.43	70°	0.9397

(b) The graph shows that a α sine of the angle. Hence g is the acceleration when the sine of the angle is 1.0000 (angle equals 90°). g turns out to be approximately 12.20 ft/sec².

(c) Mars − g at surface of Mars = 0.38(32) = 12.16 ft/sec²



Question 5–5–1

(a) $$R = \sqrt[3]{\frac{T^2 G\ M}{4\pi^2}}$$

$$R = \sqrt[3]{\frac{(2.5514 \times 10^6)^2(6.670 \times 10^{-11})(6 \times 10^{24})}{4(3.14)^2(80)}}$$

$$R \approx \sqrt[3]{\frac{260.384 \times 10^{25}}{3155}}$$

$$R \approx \sqrt[3]{0.8253 \times 10^{24}}$$

$\log 0.8253 = 9.91661 - 10$

$\log 0.8253 = 29.91661 - 30$

$\log \sqrt[3]{0.8253} = 9.97220 - 10$

$\sqrt[3]{0.8253} = 0.9380$

$\sqrt[3]{10^{24}} = 10^8$

$R \approx 9.380 \times 10^7$ meters (58,260 miles)

(b) $$V = \sqrt{\frac{GM}{R}}$$

$$V = \sqrt{\frac{(6.670 \times 10^{-11})(6 \times 10^{24})}{(9.380 \times 10^7)80}}$$

$$V \approx \sqrt{\frac{40 \times 10^{13}}{750.4 \times 10^7}}$$

$V \approx \sqrt{5.33 \times 10^4}$

$V \approx 231$ m/sec. (516.5 mi/hr)

Question 5–5–2

$$T = 2\pi \sqrt{\frac{R^3}{GM}}$$

$$T = 2(3.14) \sqrt{\frac{(7.819 \times 10^6)^3}{(6.670 \times 10^{-11})(6 \times 10^{24})}}$$

$$T \approx 6.28 \sqrt{\frac{4.78 \times 10^{20}}{40 \times 10^{13}}}$$

$$T \approx 6.28 \sqrt{1.195 \times 10^6}$$

$$T \approx 6.28(1.09 \times 10^3)$$

$$T \approx 6845 \text{ seconds } (114.08 \text{ minutes})$$

Question 5–5–3

$$F_1 = G\frac{m_1M}{R_1^2}$$

$$F_1 = \frac{m_1V_1^2}{R_1}$$

$$G\frac{m_1M}{R_1^2} = \frac{m_1V_1^2}{R_1}$$

$$GM = V_1^2R_1$$

$$V_1 = \frac{2\pi R_1}{T_1}$$

$$V_1^2 = \frac{4\pi^2R_1^2}{T_1^2}$$

$$GM = \frac{4\pi^2R_1^3}{T_1^2} \quad \text{so} \quad GM = \frac{4\pi^2R_2^3}{T_2^2}$$

$$\frac{4\pi^2R_1^3}{T_1^2} = \frac{4\pi^2R_2^3}{T_2^2}$$

$$T_1^2R_2^3 = T_2^2R_1^3$$

$$\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3}$$

Question 5–6–1

R_1 = the initial distance the spacecraft is from the center of the earth.

$V_1 = \sqrt{\frac{GM}{R_1}}$ = the velocity of the spacecraft at distance R_1.

R_2 = the final distance the spacecraft is from the center of the earth.

$V_2 = \sqrt{\frac{GM}{R_2}}$ = the velocity of the spacecraft at distance R_2.

$$W = E_{\text{total}} \text{ (initial)} - E_{\text{total}} \text{ (final)}$$

$$W = \left[GmM\left(\frac{1}{r} - \frac{1}{R_1}\right) + \tfrac{1}{2}mV_1^2\right] - \left[GmM\left(\frac{1}{r} - \frac{1}{R_2}\right) + \tfrac{1}{2}mV_2^2\right]$$

$$W = \left[GmM\left(\frac{1}{r} - \frac{1}{R_1}\right) + \tfrac{1}{2}m\frac{GM}{R_1}\right] - \left[GmM\left(\frac{1}{r} - \frac{1}{R_2}\right) + \tfrac{1}{2}m\frac{GM}{R_2}\right]$$

$$W = \left[\frac{GmM}{r} - \frac{GmM}{R_1} + \tfrac{1}{2}\frac{GmM}{R_1}\right] - \left[\frac{GmM}{r} - \frac{GmM}{R_2} + \tfrac{1}{2}\frac{GmM}{R_2}\right]$$

$$W = \frac{GmM}{r} - \frac{GmM}{R_1} + \tfrac{1}{2}\frac{GmM}{R_1} - \frac{GmM}{r} + \frac{GmM}{R_2} - \tfrac{1}{2}\frac{GmM}{R_2}$$

$$W = \tfrac{1}{2}\frac{GmM}{R_2} - \tfrac{1}{2}\frac{GmM}{R_1}$$
$$W = \frac{GmM}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$
$$W = \frac{(6.670 \times 10^{-11})(8.8 \times 10^{3})(6 \times 10^{24})}{2}\left[\frac{1}{6.99985 \times 10^{6}} - \frac{1}{7.016 \times 10^{6}}\right]$$
$$W \approx 176.1 \times 10^{16}\,[(0.1429 \times 10^{-6}) - (0.1425 \times 10^{-6})]$$
$$W \approx (176.1 \times 10^{16})(4 \times 10^{-10})$$
$$W \approx 7.044 \times 10^{8} \text{ joules } (5.283 \times 10^{8} \text{ ft(lbs)}).$$

Question 5–6–2

(a) R_2 becomes infinitely large as $\frac{R_1V^2}{2GM}$ approaches 1.

(b) $$\frac{R_1V^2}{2GM} = 1$$
$$R_1V^2 = 2GM$$
$$V = \sqrt{\frac{2GM}{R_1}} \quad \text{(the escape velocity)}$$

Question 5–6–3

(a) $$V = \sqrt{\frac{2GM}{R}}$$
$$V = \sqrt{\frac{2(6.670 \times 10^{-11})(6 \times 10^{24})}{6.8545 \times 10^{6}}}$$
$$V \approx \sqrt{\frac{80.04 \times 10^{13}}{6.8545 \times 10^{6}}}$$
$$V \approx \sqrt{1.168 \times 10^{8}}$$
$$V \approx 10{,}800 \text{ m/sec. } (35{,}424 \text{ ft/sec. or } 24{,}149 \text{ mi/hr.})$$

(b) $$V = \sqrt{\frac{80.04 \times 10^{13}}{7.662 \times 10^{6}}}$$
$$V \approx \sqrt{1.045 \times 10^{8}}$$
$$V \approx 1.02 \times 10^{4}$$
$$V \approx 10{,}200 \text{ m/sec. } (33{,}456 \text{ ft/sec. or } 22{,}807 \text{ mi/hr})$$

Question 5–6–4

$$R = \sqrt[3]{\frac{T^2GM}{4\pi^2}}$$
$$R = \sqrt[3]{\frac{(6.6 \times 10^{3})^2(6.670 \times 10^{-11})(6 \times 10^{24})}{4(3.14)^2}}$$
$$R \approx \sqrt[3]{\frac{17.424 \times 10^{21}}{39.44}}$$
$$R \approx \sqrt[3]{0.4418 \times 10^{21}}$$
$$\log 0.4418 = 9.64523 - 10$$
$$\log 0.4418 = 29.64523 - 30$$
$$\log \sqrt[3]{0.4418} = 9.88174 - 10$$
$$\sqrt[3]{0.4418} = 0.7616$$
$$\sqrt[3]{10^{21}} = 10^{7}$$

$$R = 7.616 \times 10^6 \text{ meters}$$

$$V = \sqrt{\frac{GM}{R}}$$

$$V = \sqrt{\frac{(6.670 \times 10^{-11})(6 \times 10^{24})}{7.616 \times 10^6}}$$

$$V \approx \sqrt{\frac{40.02 \times 10^{13}}{7.616 \times 10^6}}$$

$$V \approx \sqrt{52.55 \times 10^6}$$

$$V \approx 7{,}250 \text{ m/sec.}$$

$$\text{Circular orbital velocity} = \sqrt{\frac{GM}{R}}$$

$$\text{Escape velocity} = \sqrt{\frac{2GM}{R}}$$

Escape velocity = $\sqrt{2}$ times the circular orbital velocity.

$$V_e \approx \sqrt{2}\,(7{,}250)$$

$$V_e \approx 10{,}252 \text{ m/sec.}$$

Additional velocity needed is 10,252–7,250 = 3,002 m/sec. (9,847 ft/sec., or 6,712 mi/hr.)

Question 5–7–1

$$\frac{V_p}{V_a} = \frac{1 + e}{1 - e}$$

$$V_a + V_a e = V_p - V_p e$$

$$V_p e + V_a e = V_p - V_a$$

$$e(V_p + V_a) = V_p - V_a$$

$$e = \frac{V_p - V_a}{V_p + V_a}$$

Question 5–7–2

$$E = E_k + E_p$$

$$E = \tfrac{1}{2}mV^2 + \left[GmM\left(\frac{1}{r} - \frac{1}{a}\right)\right]$$

$$V = \sqrt{\frac{GM}{a}}$$

$$V^2 = \frac{GM}{a}$$

$$E = \tfrac{1}{2}\frac{GmM}{a} + \frac{GmM}{r} - \frac{GmM}{a}$$

$$E = \frac{GmM}{r} - \tfrac{1}{2}\frac{GmM}{a}$$

$$E = GmM\left(\frac{1}{r} - \frac{1}{2a}\right)$$

Both have the same energy.

Question 5–7–3

$$V^2 = GM\left(\frac{2}{R} - \frac{1}{a}\right)$$

$$R_p = a - ae$$

$$R_a = ae + ae + (a - ae) = a + ae$$

$$V_p = \sqrt{GM\left(\frac{2}{a - ae} - \frac{1}{a}\right)}$$

$$V_a = \sqrt{GM\left(\frac{2}{a + ae} - \frac{1}{a}\right)}$$

$$\text{Momentum at perigee} = m\sqrt{GM\left(\frac{2}{a - ae} - \frac{1}{a}\right)}(a - ae)$$

$$\text{Momentum at apogee} = m\sqrt{GM\left(\frac{2}{a + ae} - \frac{1}{a}\right)}(a + ae)$$

$$m\sqrt{GM\left(\frac{2}{a - ae} - \frac{1}{a}\right)}\ (a - ae) = m\sqrt{GM\left(\frac{2}{a + ae} - \frac{1}{a}\right)}\ (a + ae)$$

$$GM\left(\frac{2}{a - ae} - \frac{1}{a}\right)(a - ae)^2 = GM\left(\frac{2}{a + ae} - \frac{1}{a}\right)(a + ae)^2$$

$$\frac{2(a - ae)^2}{(a - ae)} - \frac{(a - ae)^2}{a} = \frac{2(a + ae)^2}{(a + ae)} - \frac{(a + ae)^2}{a}$$

$$\frac{2a(a - ae) - (a - ae)^2}{a} = \frac{2a(a + ae) - (a + ae)^2}{a}$$

$$2a^2 - 2a^2e - (a^2 - 2a^2e + a^2e^2) = 2a^2 + 2a^2e - (a^2 + 2a^2e + a^2e^2)$$

$$2a^2 - 2a^2e - a^2 + 2a^2e - a^2e^2 = 2a^2 + 2a^2e - a^2 - 2a^2e - a^2e^2$$

$$a^2 - a^2e^2 = a^2 - a^2e^2$$

$$1 - e^2 = 1 - e^2$$

$$1 = 1$$

A satellite has the same angular momentum at apogee and perigee. Therefore, it appears that angular momentum is conserved.

Question 5–7–4

(a) $V^2 = GM\left(\frac{2}{R} - \frac{1}{a}\right)$

$$a = \frac{(1.037 \times 10^6) + (1.318 \times 10^6) + 2(6.37 \times 10^6)}{2}$$

$$a = \frac{15.095 \times 10^6}{2} = 7.548 \times 10^6 \text{ meters}$$

$$V_p^2 = (6.670 \times 10^{-11})(6 \times 10^{24})\left[\frac{2}{7.407 \times 10^6} - \frac{1}{7.548 \times 10^6}\right]$$

$$V_p^2 \approx 40.02 \times 10^{13}\,[(0.27 \times 10^{-6}) - (0.1325 \times 10^{-6})]$$

$$V_p^2 \approx 40.02 \times 10^{13}\,(0.1375 \times 10^{-6})$$

$$V_p^2 \approx 5.50 \times 10^7$$

$$V_p \approx \sqrt{0.55 \times 10^8}$$

$V_p \approx$ 7,420 m/sec (24,338 ft/sec or 16,591 mi/hr)

$$V_a^2 = 40.02 \times 10^{13}\left[\frac{2}{7.688 \times 10^6} - \frac{1}{7.548 \times 10^6}\right]$$

$$V_a^2 \approx 40.02 \times 10^{13}\,[(0.2601 \times 10^{-6}) - (0.1325 \times 10^{-6})]$$

$$V_a^2 \approx 40.02 \times 10^{13}\,(0.1276 \times 10^{-6})$$

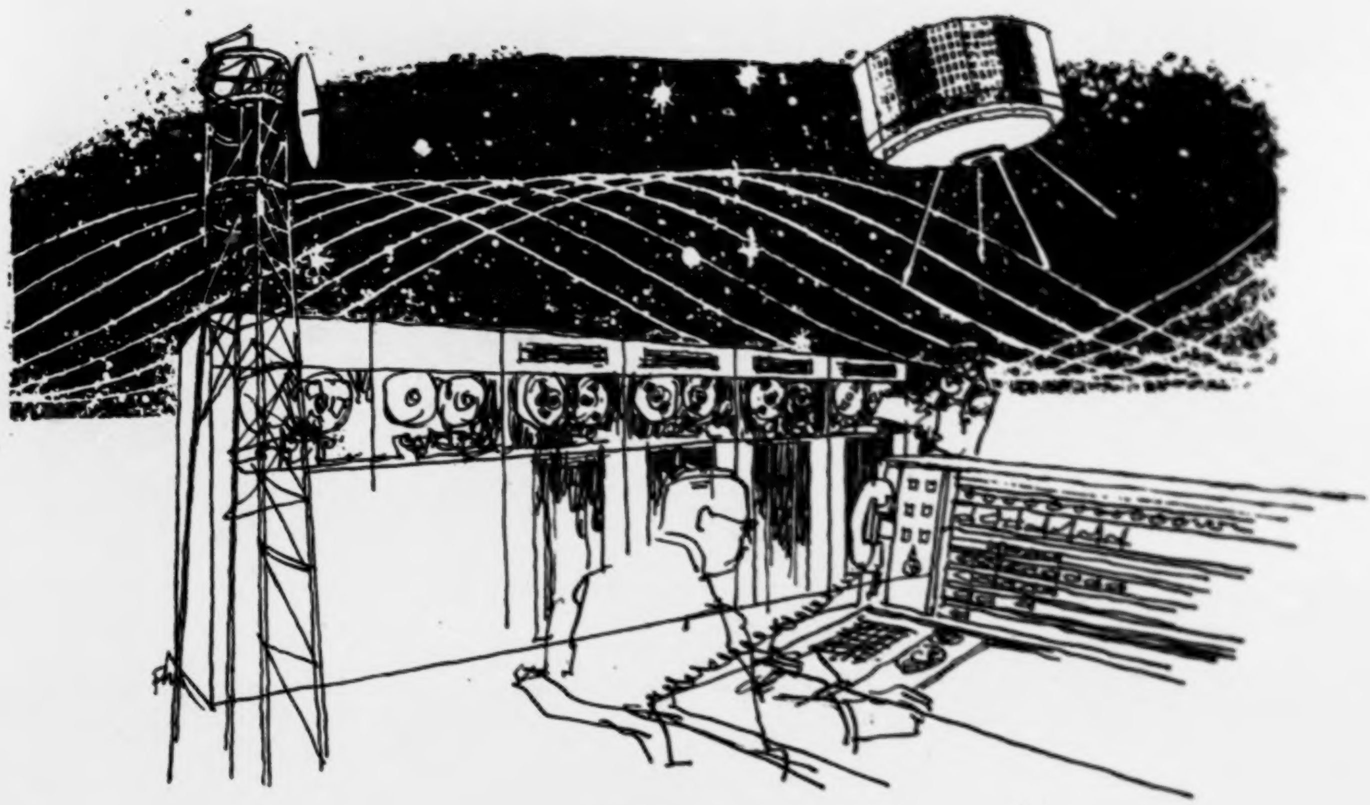
$$V_a^2 \approx 5.11 \times 10^7$$

$$V_a \approx \sqrt{0.511 \times 10^8}$$

$V_a \approx$ 7,150 m/sec (23,452 ft/sec or 15,987 mi/hr)

(b)
$$e = \frac{V_p - V_a}{V_p + V_a}$$
$$e \approx \frac{7{,}420 - 7{,}150}{7{,}420 + 7{,}150}$$
$$e \approx \frac{270}{14{,}570}$$
$$e \approx 0.019$$

(c)
$$T^2 = \frac{4\pi^2 a^3}{GM}$$
$$T^2 \approx \frac{4(3.14)^2(7.548 \times 10^6)^3}{(6.670 \times 10^{-11})(6 \times 10^{24})}$$
$$T^2 \approx \frac{39.44(430 \times 10^{18})}{40.02 \times 10^{13}}$$
$$T^2 \approx \frac{16959 \times 10^{18}}{40.02 \times 10^{13}}$$
$$T^2 \approx 42.38 \times 10^6$$
$$T \approx \sqrt{42.38 \times 10^6}$$
$$T \approx 6{,}510 \text{ seconds (108.5 minutes)}$$



Chapter 6

COMPUTERS ARE NEEDED

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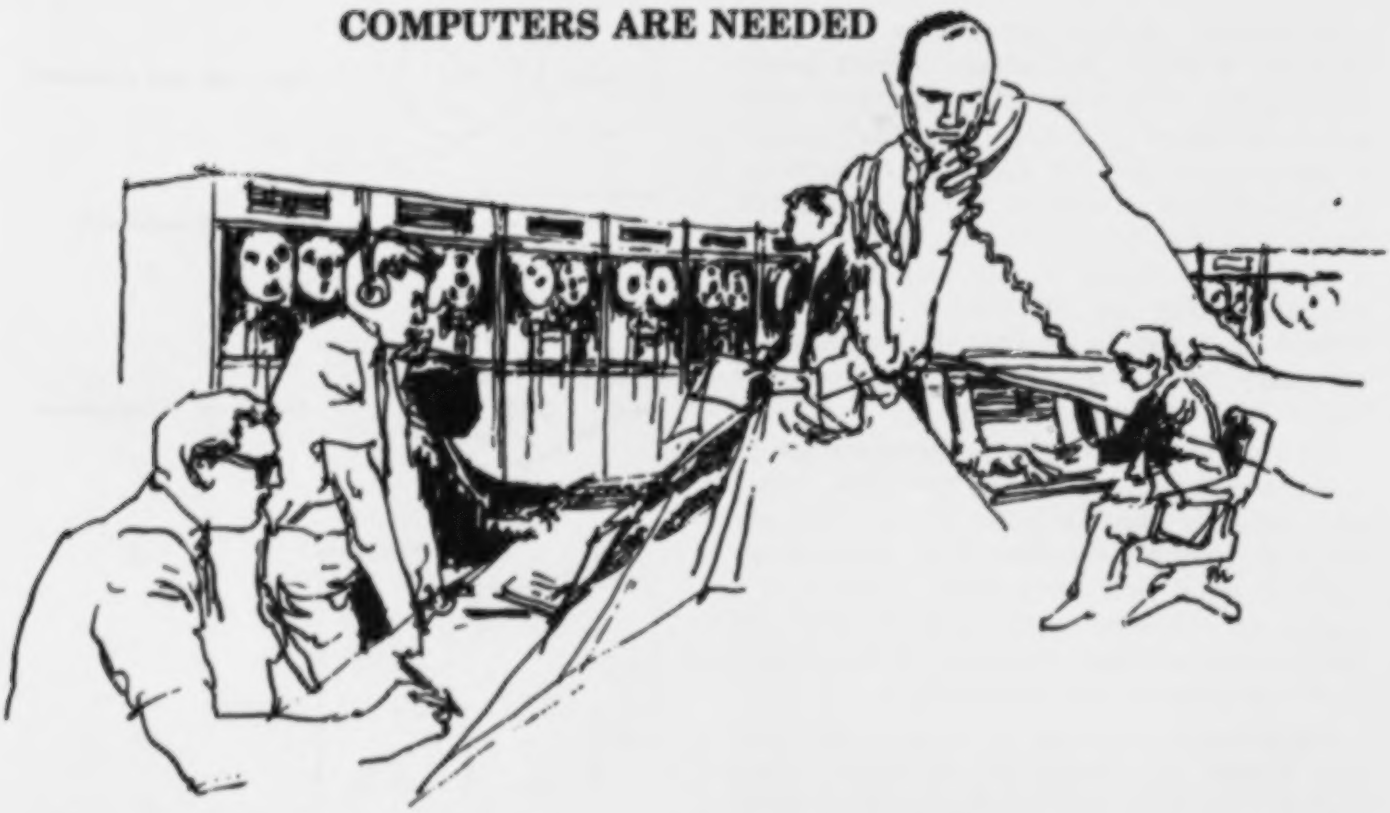
COMPUTERS ARE NEEDED



This chapter is concerned with an important new tool in space research: the computer and computer programing. However readers without experience in this area will also gain much from this chapter. There will be many calculations to perform and often the arithmetical process is repetitive. This is the type of mathematics that is monotonous for students to do but the type that can be easily done by electronic digital computers.

The programs are written in the computer languages Gotran and Afit Fortran. Afit Fortran is similar to the many other versions of Fortran but it has its own peculiarities. Therefore, persons who are using other Fortran languages are not expected to use the programs without making some modifications. Anyone familiar with other machine languages such as Algol and Fortran I or II will find that most of the work in the chapter is done in general terms. The emphasis upon flow charts should make the work easier for readers with varied backgrounds to comprehend and apply.

6–1 Computers are Essential

Much of the progress that has been made in space exploration during the past few years and much of the progress that will be made in years to come will be attributed to digital computers. Certainly without the use of computers, it would have been difficult to embark on the space program. Only because of rapid calculations, high degrees of accuracy, reliability, and the large storage capacities of present day digital computers, have these orbital missions been possible.

NASA's, Goddard Space Flight Center at Greenbelt, Maryland, is the "hub" of the space agency's world wide tracking activities and its computer complex is the very heartbeat of these globe-circling efforts. Computers help guide manned and scientific satellites on their flights, whether orbiting the earth or a far-away planet. Computers also reduce new-found data recorded on these missions into facts and figures for study and evaluation by the experimenters. On an average day the Goddard computers record about 60 miles of magnetic tape data!

Let's consider some of the functions of one of the computers used. When an astronaut is in orbit, a computer is used to chart his exact course—in "real-time"—virtually instantaneously! Thus during the launching of the capsule the computer is calculating launch trajectory, insertion parameters, and landing point. During the orbiting of the capsule, the computer is calculating the capsule position, orbital parameters, retro-fire time for re-entry, and actual impact point. The rapidity with which this computer operates is indicated by the fact that each second it can perform 250,000 additions or subtractions of numbers with numerals having ten decimal digits. Multiplication and division are somewhat slower; it can *only* do 100,000 multiplications or 62,500 divisions per second. It is easy to see that the calculations necessary for world-wide tracking would be impossible without computers.

To emphasize how vital computers are to space exploration, consider the data being sent back by the Explorer VI, a relatively small and simple satellite. The information received from this *one satellite would take a staff of five thousand people working forty-eight hours a week one year to process!* A large computer can process the same data in 1.2 days.

Digital computers are so reliable that they can almost be considered errorless. Since digital computers can be programed to check the arithmetical operations that they perform, the errors that occur are nearly always a result of the program and not the machine.

6–2 Flow Charts

A flow chart is a schematic diagram that is used as an aid in computer programing. The diagram shows the procedures to be used and the sequence of steps that must be followed to arrive at the solution of a problem. Students are given much freedom in making flow charts since any diagram that is helpful in preparing a program might be considered to be a flow chart. For uniformity, the same symbolism will be used throughout this chapter to indicate a specific process in programing. This symbolism is illustrated in Figure 6–1.

Because of the many different ways that programers have to represent branch statements and do loops in flow charts, let us examine closely these diagrams in Figure 6–1.

In the branch statement, suppose DISC = $B^2 - 4AC$. When DISC > 0, a branch is made to a specific location in the program. Likewise, branches are made to other locations whenever DISC = 0 and DISC < 0.

The *do* loop in Figure 6–1 indicates that calculations must be performed, K is incremented (increased) by one and the calculations are performed again. This process continues for $K \leq 5$ but as soon as $K > 5$, an exit is made from the do loop to a designated location in the program.

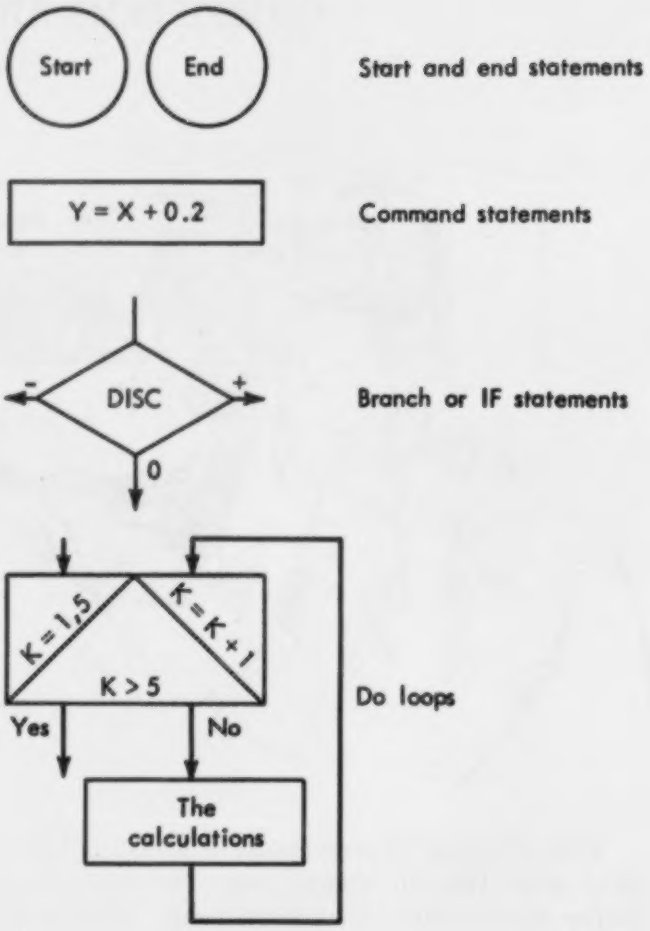


Figure 6–1

A programer may write many programs without ever making a flow chart, but as the problems become more complex, it becomes essential to use flow charts. Only by using such devices can the programer gain the proper perspective for the whole problem. Therefore, it is strongly suggested that students make a flow chart for every program, no matter how simple. Then when complex problems do arise, the programs will be much easier to write.

We take the problem of finding coordinates for points of an ellipse as our first example of flow charting and machine computation. Remember that planets have elliptical paths around the sun; the moon has an elliptical path around Earth; most satellites have elliptical paths around Earth.

The usual equation for an ellipse with center at the origin, major axis along the x-axis,

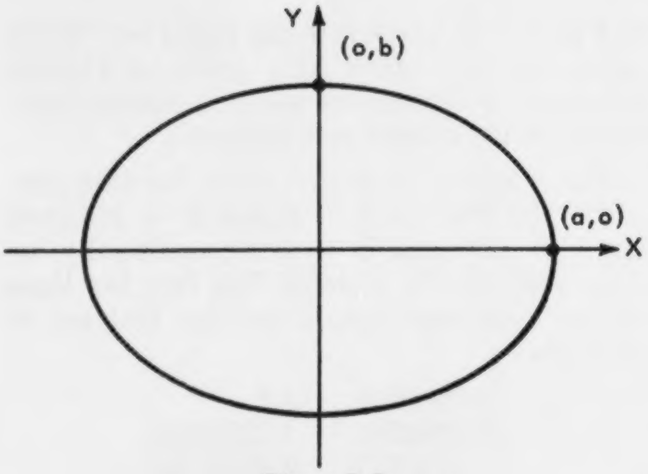


Figure 6-2

major axis 2a, and minor axis 2b (Figure 6–2) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

For work on a computer it is customary to use capital letters. We shall also use the functions sine and cosine of a variable T (Section 1–7). These functions have the special property

$$\sin^2 \theta + \cos^2 \theta = 1$$

for any real number θ. Thus if

$$X = A \cos (T)$$
$$Y = B \sin (T)$$

then
$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1$$

and the variable T may be used to identify points of the ellipse. For this reason T is called a *parameter* and the two equations expressing X and Y in terms of T are *parametric equations* of the ellipse. If $A > B$, then 2A is the length of the major axis and 2B is the length of the minor axis of the ellipse. If $A = B$, then the graph of the equations is a circle of radius A.

A flow chart is given in Figure 6–3 for a program that will print ordered pairs (X, Y) of coordinates for the set of points of the graph of the equations as the parameter T increases with increments of 0.3 from 0 to 6.3 radians. Remember that 2π radians equals 1 revolution and $2\pi \approx 6.3$.

Once the coordinates (X, Y) for 6.3 radians have been printed, then the program will initialize and read a new set of data for another ellipse.

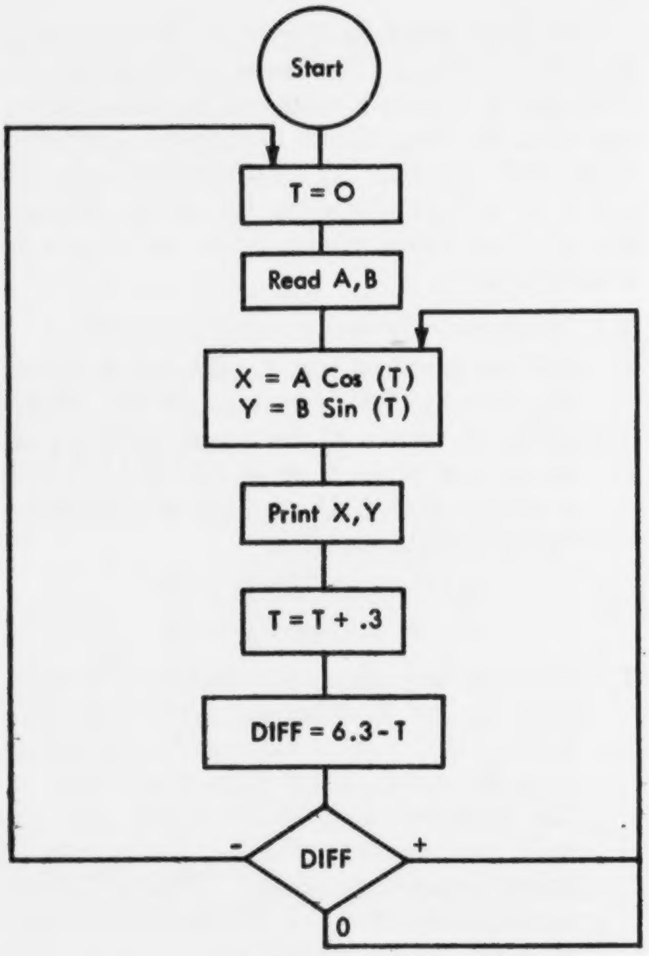


Figure 6-3

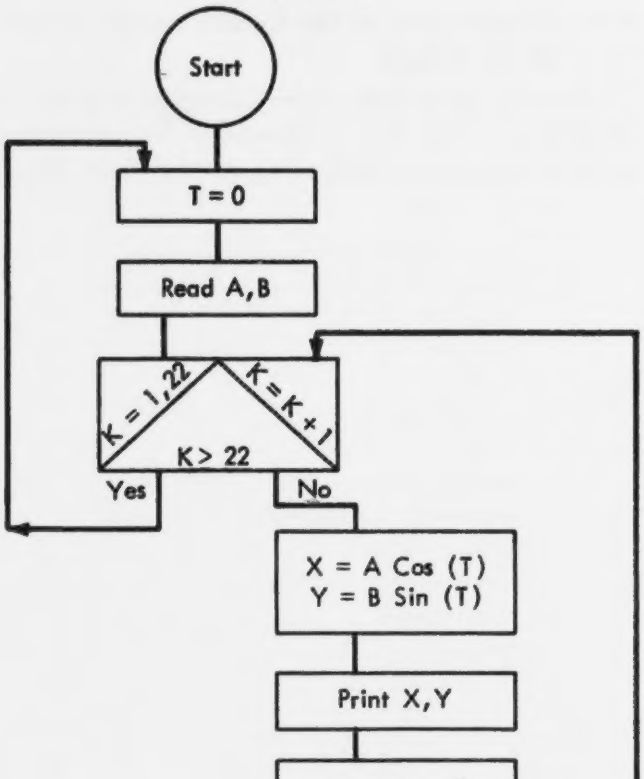


Figure 6-4

The flow chart in Figure 6-3 involves a branch command. The same problem can be done and the same results can be obtained by use of a do loop. Since twenty-two ordered pairs will be printed, instructions can be given to obtain this data by going through the do loop twenty-two times as shown in Figure 6-4.

6-1 Exercises Computers Are Essential

1. For the problem just illustrated concerning coordinates of points on an ellipse given in terms of its parametric equations, can A = 0 or B = 0? If this is possible, sketch the graph of the parametric equations when

   (a) $A = 0$ and $B \neq 0$
   (b) $A = 0$ and $B = 0$

2. Make a flow chart for finding the solutions X_1 and X_2 of the quadratic equation $AX^2 + BX + C = 0$ for different values of A, B, and C. If for a given set of data, the equation has roots which are not real, instructions should be given to print "imaginary roots." What will your program do if A = 0? If A = 0 and B = 0?

6-3 Coordinates of the Points on the Graph of an Ellipse

As soon as a flow chart is completed for a problem, then the programer is ready to write a program. Referring to the flow chart in Figure 6-3, a program can easily be written using the flow chart as a guide. A Gotran program, which makes use of a branch command, is illustrated in Figure 6-5.

```
C     GOTRAN PROGRAM USING THE BRANCH COMMAND
C     COORDINANTS OF ELLIPSE USING PARAMETRIC EQUATIONS
    1 T=0.0
      READ,A,B
    2 X1=COS(T)
      X=A*X1
      Y1=SIN(T)
      Y=B*Y1
      PRINT,X,Y
      T=T+0.3
      DIFF=6.3-T
      IF (DIFF)1,2,2
      END
9.0
16.0
4200.0
5000.0
4400.0
4400.0
```

Figure 6-5

Three sets of data are given for this program, the first for A = 9 and B = 16; then for A = 4200 and B = 4000; and finally for A = 4400 and B = 4400. The first few lines of the computer output for the first set of data are:

```
(9.0000000,    0.0        )
(8.5980284,    4.7283232)
(7.4280205,    9.0342795)
(5.5944897,   12.533230 )
(3.2612198,   14.912625 )
```

The complete set of ordered pairs is plotted in Figure 6-6.

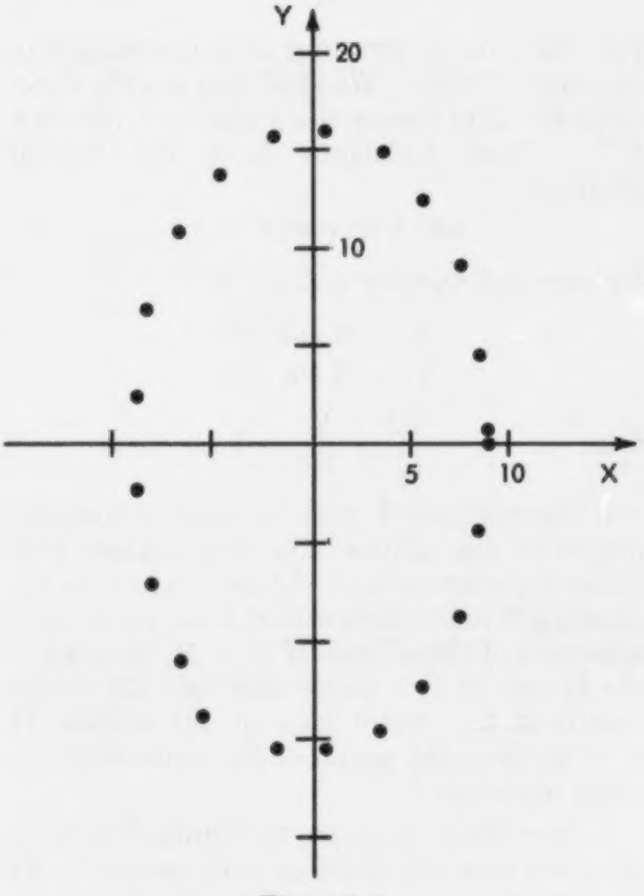


Figure 6-6

The flow chart in Figure 6-4 makes use of a do loop to print ordered pairs of points on the graph of the ellipse. A Fortran program based on this flow chart is shown in Figure 6-7.

For comparison with the output of the previous program, the first few lines of out-

```
C       FORTRAN PROGRAM USING THE DO LOOP
C       COORDINANTS OF ELLIPSE USING PARAMETRIC EQUATIONS
      3 T=0.0
        READ,A,B
        DO 1 K=1,22
        X=A*COS(T)
        Y=B*SIN(T)
        PRINT 2,X,Y
      2 FORMAT(/2F10.3)
      1 T=T+0.3
        GO TO 3
        END
9.0, 16.0
4200.0, 5000.0
4400.0, 4400.0
```

Figure 6–7

put of this program are:

```
(9.000,    .000)
(8.598,   4.728)
(7.428,   9.034)
(5.594,  12.533)
(3.261,  14.912)
```

Notice that in the Fortran output, the digits beyond three places to the right of the decimal point were merely dropped off and no method of rounding-off was employed. Gotran answers are all printed with eight decimal digits; Afit Fortran answers are printed as specified in the program with a maximum of eight digits. It is left to the programer to determine the number of significant digits in the answer, keeping in mind that the answer can not be expected to contain more significant digits than were present in the least accurate element of the data that was used.

6–3 Exercises Coordinates of the Points on the Graph of an Ellipse

1. Refer to Exercise 2 in Section 6–2 and write a program that will print the real roots of the quadratic equation.
2. Use the plot statement in Gotran and write a program that will have the computer graph the equation

$$Y = \sqrt{2304 - 9X^2}$$

with increments of 1.0 for X. Notice that these points are points of an ellipse with the major axis along the Y-axis, $B^2 = 258$ and $A^2 = 2304$.

3. In Section 3–6, it was shown how to determine the percentage of earth that is visible for a satellite at different elevations. Write a program that will find the area of Earth that is visible from an elevation of 200 miles; determine what percent this area is of the total area of Earth; and have the program print this data for increments of 200 miles up to a 5000 mile elevation.
4. As a slight variation of Exercise 3, write a program that will punch out data on cards for increments of 200 miles up to an elevation of 5000 miles and at this point change the increment to 1000 miles and punch out data for elevations up to 25000 miles.

6–4 Area Under a Curve

As mentioned in Chapter 5, it is often necessary to find the area under a curve. Computers may be used to find the area to a high degree of accuracy without using formulas derived from calculus.

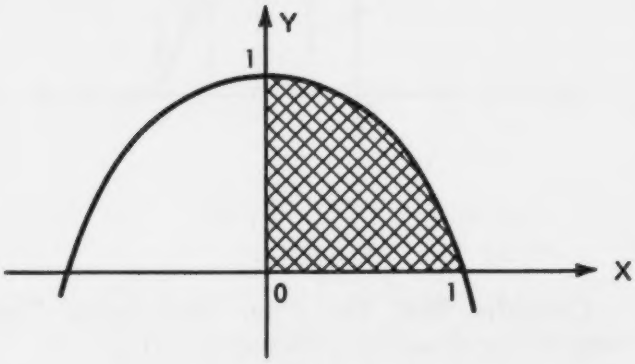


Figure 6–8

Consider the problem of finding the area in the first quadrant bounded by the X-axis, the Y-axis, and the graph of the equation $Y = 1 - X^2$ (Figure 6–8). Notice that the values of X for the desired region are $0 \leq X \leq 1$. We divide this interval of the X-axis into N equal parts, consider the corresponding values for X

$$0, \frac{1}{N}, \frac{2}{N}, \frac{3}{N}, \ldots, \frac{N-1}{N}, 1$$

and compute the corresponding values of Y. For N = 4 we have

$$0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$$

for X and

$$1, \frac{15}{16}, \frac{3}{4}, \frac{7}{16}, 0$$

for Y. Then we consider inner rectangles as in Figure 6–9 and outer rectangles as in Figure 6–10.

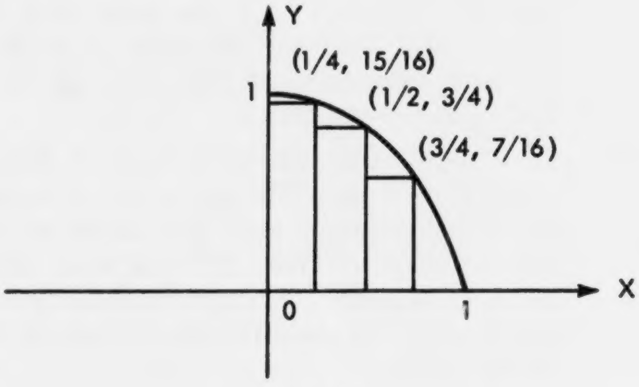


Figure 6–9

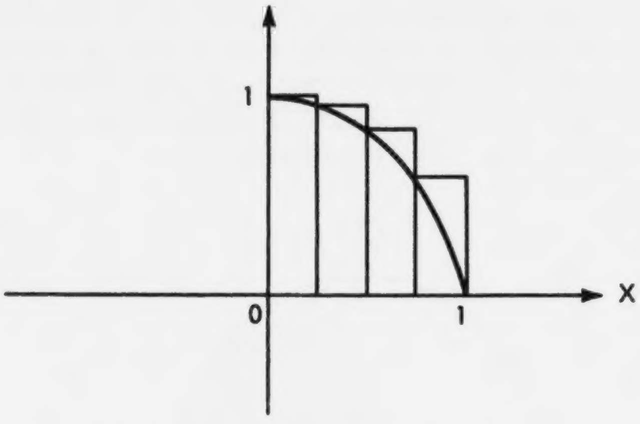


Figure 6–10

Consider first the inner rectangles. The sum of the areas may be expressed as

$$\left(\frac{1}{4}\times\frac{15}{16}\right)+\left(\frac{1}{4}\times\frac{3}{4}\right)+\left(\frac{1}{4}\times\frac{7}{16}\right)$$
$$=\frac{1}{4}\left(\frac{15}{16}+\frac{3}{4}+\frac{7}{16}\right)=\frac{17}{32};$$

that is, we may find the sum of the areas by multiplying the common width of the rectangles by the sum of the values of the Y-coordinates (ordinates) corresponding to the values of X starting with $X = \frac{1}{N}$.

Now consider the outer rectangles. The sum of the areas may be expressed as

$$\left(\frac{1}{4}\times 1\right)+\left(\frac{1}{4}\times\frac{15}{16}\right)+\left(\frac{1}{4}\times\frac{3}{4}\right)+$$
$$\left(\frac{1}{4}\times\frac{7}{16}\right)=\frac{1}{4}\left(1+\frac{15}{16}+\frac{3}{4}+\frac{7}{16}\right)=\frac{25}{32};$$

that is, we may find the sum of the areas by multiplying the common width of the rectangles by the sum of the values of the Y-coordinates corresponding to the values of X starting with X = 0.

The desired area A satisfies the relation

$$\frac{17}{32} < A < \frac{25}{32}.$$

The average value,

$$\frac{1}{2}\left(\frac{17}{32}+\frac{25}{32}\right)=\frac{1}{2}\times\frac{42}{32}=\frac{21}{32}$$

may be used as an approximation for A that is better than either of the sums of areas of rectangles. The effectiveness of this method, even with N = 4, can be observed by comparing 21/32 with the actual area 2/3 as may be found using calculus. Still better approximations may be found by using larger values of N.

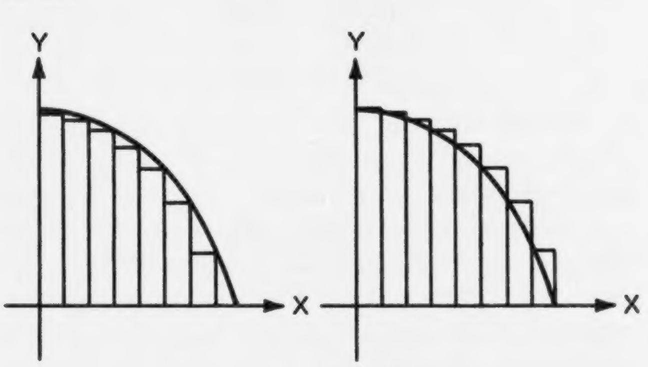


Figure 6–11

The inner and outer rectangles for N = 8 are shown in Figure 6–11. Then the sum of the areas of the inner rectangles is S_1 where

$$S_1=\frac{1}{8}\left(\frac{63}{64}+\frac{60}{64}+\frac{55}{64}+\frac{48}{64}+\right.$$
$$\left.\frac{39}{64}+\frac{28}{64}+\frac{15}{64}\right)=\frac{77}{128}.$$

The sum of the areas of the outer rectangles is S_2 where

$$S_2=\frac{1}{8}\left(1+\frac{63}{64}+\frac{60}{64}+\frac{55}{64}+\right.$$
$$\left.\frac{48}{64}+\frac{39}{64}+\frac{28}{64}+\frac{15}{64}\right)=\frac{93}{128}.$$

The average of these two areas,

$$\frac{1}{2}\left(\frac{77}{128}+\frac{93}{128}\right)=\frac{85}{128},$$

differs from the actual area by only 1 part in 384.

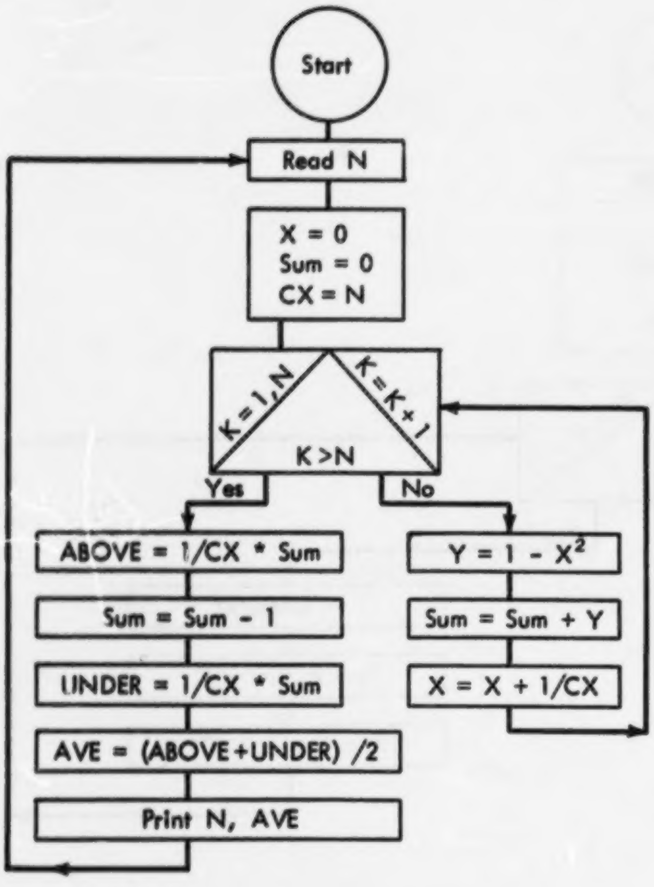


Figure 6-12

A flow chart for the procedure that we have been using is given in Figure 6-12. Notice that in the flow chart the sum of the areas of the outer rectangles is found first, then the value 1 of Y when X = 0 is subtracted from the sum of the Y-values (ordinates) and the sum of the areas of the inner rectangles is found. Finally, the average of the two areas is found. A Fortran program for estimating the area is given in Figure 6-13.

```
C     FORTRAN PROGRAM FOR AREA UNDER Y = 1 - X2
    1 READ,N
      X=0.0
      SUM=0.0
      CX=N
      DO 2 K=1,N
      SUM=SUM+1.0-X*X
    2 X=X+1.0/CX
      ABOVE=1.0/CX*SUM
      UNDER=1.0/CX*(SUM-1.0)
      AVE=(ABOVE+UNDER)/2.0
      PRINT 3,N,AVE
    3 FORMAT(/I4,F10.6)
      GO TO 1
      END
  4   ____________
  8
  25
  50
  100
```

Figure 6-13

Remember that the actual area under the curve is ⅔ and observe the high degree of accuracy in the computer output for increasing values of N:

N	Area
4	.656250
8	.664062
25	.666400
50	.666600
100	.666650

Now consider the problem of determining the useful work that is done to get the payload of the Scout rocket out to an elevation of 1000 miles. As was explained in Section 5-4, this is equivalent to finding the area under the curve $F = \frac{GMm}{R^2}$ as R is displaced from 4000 miles to 5000 miles. If we assume the weight of the payload to be 500 pounds, then to find the work in foot pounds, the masses of the earth and the payload must be changed to slugs and R changed to feet. If the weight of the earth is 1.32×10^{25} pounds and $G = 3.41 \times 10^{-8}$ ft^4/(lb sec^4), then a program can be written to find the area under the curve by taking N equal intervals between R = 4000 and R = 5000 and by finding areas of rectangles as was done in our previous example. If N becomes large, then the sum of the areas of the rectangles approaches the area under the curve. A flow chart for this problem is given in Figure 6-14.

Notice that in the flow chart N is initially 50 and then is incremented by 50 until it becomes 150. The program will have the area (that is, the work that is done) printed out along with the corresponding value of N. After this data has been printed for N = 150, then the accuracy of the results may be checked by calculating the work by the formula $W = GMm(1/r - 1/R)$, which is derived from calculus. The following is a listing of the computer output for this problem.

N	Work
50	20555435.E + 02
100	20555274.E + 02
150	20555243.E + 02
	20555244.E + 02

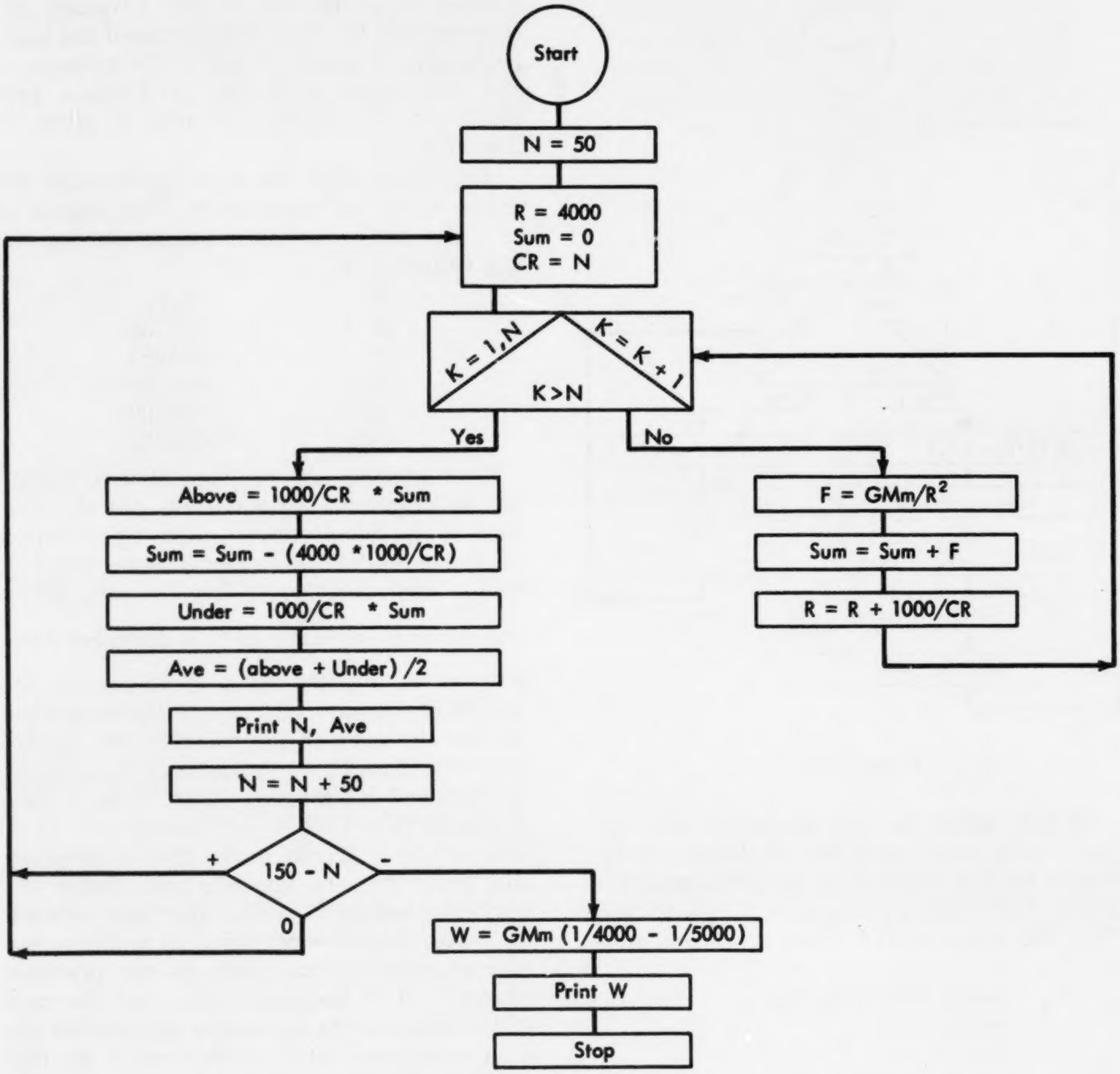

Start
N = 50
R = 4000
Sum = 0
CR = N
K = 1,N
K = K + 1
K>N
Yes
No
Above = 1000/CR * Sum
Sum = Sum - (4000 *1000/CR)
Under = 1000/CR * Sum
Ave = (above + Under) /2
Print N, Ave
N = N + 50
150 - N
+
-
0
W = GMm (1/4000 - 1/5000)
Print W
Stop
F = GMm/R²
Sum = Sum + F
R = R + 1000/CR

Figure 6–14

The actual amount of work done as calculated by the formula from calculus is 20555244.E2; that is, 2,055,524,400 foot pounds. This answer for the area is accurate to seven significant digits, which happens to be much more accurate than the data used in the problem.

6–4 Exercises Area Under a Curve

1. Write a program to find the area enclosed by the graph of the ellipse whose equation is $X^2 + 4Y^2 = 16$.
2. Write a program to find the area bounded by the graphs of the curves whose equations are $4X = Y^2$ and $4Y = X^2$. (Hint: Find the area under one curve, then find the area under the other curve, and subtract.)
3. Write a program that makes use of Figure 6–14 and calculate the useful work that is done to get the Scout payload out to an elevation of 1000 miles.
4. Write a program that will find the useful work that needs to be done to get the Apollo spacecraft to the point in space where the force of gravitation from the moon and the gravitation of Earth are equal. This occurs approximately 216,000

miles from the center of Earth. Assume the weight of Apollo to be 90,000 pounds on Earth. Make your program general enough so that it can calculate the work done for different weight spacecraft going out to various elevations.

6–5 The Distance Between Earth and Mars

The orbit of Earth about the sun is very nearly circular (eccentricity 0.017) with a radius of 93 million miles. Mars has an elliptical orbit (eccentricity 0.093) as shown in Figure 6–15, with foci located at points F and S.

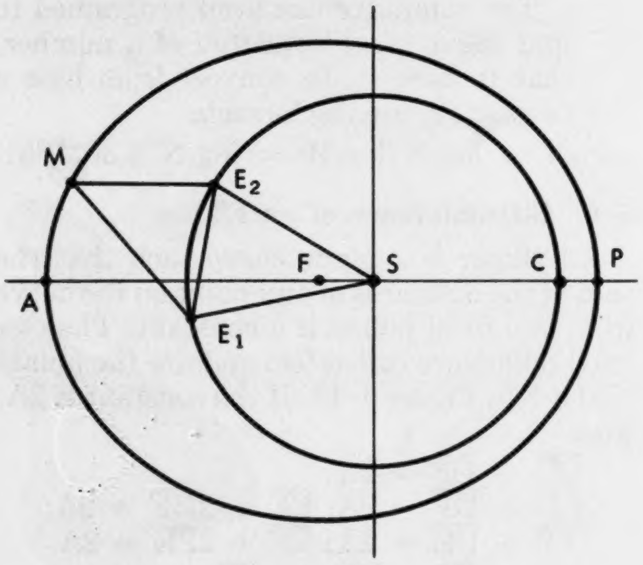


Figure 6–15

When Mars is at aphelion (A in Figure 6–15) its furthest distance from the sun, the distance from the orbit of Mars to the orbit of Earth is about 62 million miles. When Mars is at perihelion (P in Figure 6–15) the distance from the orbit of Mars to the orbit of Earth is about 35 million miles. In 1956 Mars was at perihelion and Earth was at position C in Figure 6–15. This was the last time that the two planets were in a position such that the distance between the two was a minimum. The next time that this will occur will be on August 10, 1971.

Our consideration of the distance between two planets has been in terms of straight line distances; we may think of these as "line of sight" or as the path of impulses in microwave radio transmission. These straight line distances should not be confused with the distances that one must travel to go from one planet to another. For example, the distance of 325 million miles that Mariner IV traveled from Earth to Mars was not along a straight line but rather along an arc of an elliptical orbit about the sun (Figure 6–16).

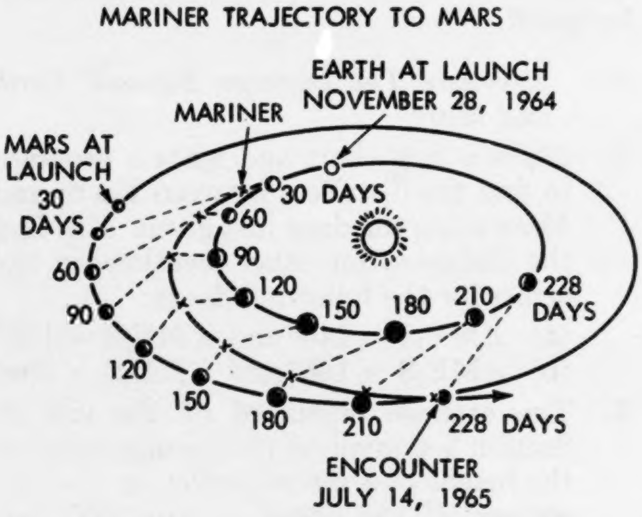


Figure 6–16

The period of time for Earth to make one revolution about the sun is 365 days; the period for Mars is 687 days. A graphical method for estimating the distance between Earth and Mars was considered in Section 3–10. To calculate this distance requires the use of the Law of Sines:

For any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Suppose that the sun is at S, Mars is in position M and Earth is in position E_1 (Figure 6–15), then $\angle ME_1S$ can be measured. When Mars is again in position M, 687 days later, Earth will be in position E_2 and $\angle ME_2S$ can be measured. The measure of $\angle E_1SE_2$ can also be found since during the 687 days that it took Mars to orbit the sun, Earth completed one orbit and 322 days of another orbit. Since Earth's orbit is nearly circular, triangle E_1SE_2 may be treated as an isosceles triangle ($\overline{E_1S} \cong \overline{E_2S}$) and $\angle E_1SE_2$ may be calculated directly. There are about 43 days of the second orbit remaining

$$\frac{43}{365} \times 360 \approx 42\tfrac{1}{2}$$

$$\angle E_1SE_2 \approx 42\tfrac{1}{2}°.$$

Then in triangle E_1SE_2, $\angle E_1 \cong \angle E_2$, and

$$\angle E_1 + \angle E_2 + 42\tfrac{1}{2}° \approx 180°$$

The measure of angles E_1 and E_2 can be found; the distance $\overline{E_1E_2}$ can be found using the Law of Sines and $\overline{E_1S} \approx 93{,}000{,}000$ miles. Suppose that by measurement $\angle ME_1S = 118°$ and $\angle ME_2S = 148°$. Then continuing our use of trigonometry the angles of triangle E_1E_2M and the distances $\overline{E_1M}$ and $\overline{E_2M}$ may be found.

6-5 *Exercises The Distance Between Earth and Mars*

1. Make a flow chart and write a program to find the distances between Earth and Mars using the data just given. Also find the distances for other locations in the orbits for the following data:
   (a) $\angle ME_1S = 140°$ and $\angle ME_2S = 110°$
   (b) $\angle ME_1S = 106°$ and $\angle ME_2S = 155°$
2. The exercise presented at the end of Section 3–5 involves the measurement of the height of a model rocket by two observers. If the observers are 1000 feet apart and A_H, A_V, B_H, and B_V are the angles that are given as discussed, then write a program that will calculate the heights H_a and H_b that are determined by the two observers. If either of the observed heights differs from the average by more than 10%, then the height is not recorded since the observations are too inaccurate. Use the data that was given in the exercise for Section 3–5. Have the computer indicate when the observations are too inaccurate.
3. The following table gives distance in parsecs, apparent magnitude, and temperature in degrees absolute for ten stars. The data compiled in the table was obtained from various sources and are considered average values.

Star	Parsecs	Apparent magnitude	Temperature
Alpha Centauri	1.32	0.3	5800.0
Sirius	2.66	−1.44	10000.0
Procyon	3.47	0.36	6500.0
Altair	5.05	0.76	8000.0
Vega	8.14	0.001	10700.0
Sun	0.0000048	−26.7	5500.0
Arcturus	11.1	−0.06	4000.0
Capella	13.7	0.9	5200.0
Aldeboran	20.8	0.78	3600.0
Regulus	25.6	1.33	13000.0

Using formulas and data from Sections 3–12, 3–13, and 3–14, write a program that will find the distance to the star in light years, the distance in astronomical units, the angle of parallax, the absolute magnitude, the luminosity in relation to the sun, and the radius of the star. Have the program print "error greater than 50%" if the angle of parallax is less than 0.01 seconds. Use 432,000 miles for the mean radius of the sun and 5500° for the temperature in degrees absolute of the sun. Use the formula $B = 2.5^{ABM}$ to show the relationship between absolute magnitude and luminosity.

The computer has been programed to find the natural logarithm of a number; that is, base e. To convert from base e to base 10, use the formula

$$\log_{10}N = \log_e N/\log_e 10 \approx \log_e N/2.3025851.$$

6–6 Circumference of an Ellipse

An ellipse is a plane curve such that the sum of the distances of any point on the curve from two fixed points is a constant. The two fixed points are called foci and are the points E and F in Figure 6–17. If the constant is 2A, then

$$\overline{EC} + \overline{CF} = 2A;$$
$$\overline{ED} + \overline{DF} = 2A;\ \overline{EF} + 2\overline{DF} = 2A;$$
$$\overline{FP} + \overline{PE} = 2A;\ \overline{EF} + 2\overline{PE} = 2A.$$

Therefore, $\overline{PE} = \overline{FD}$ and $\overline{PD} = 2A$.

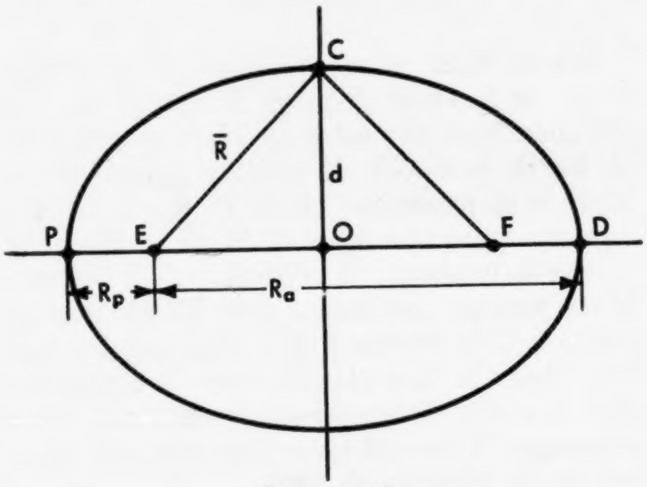


Figure 6–17

If E, which represents the center of Earth, is a focus of an elliptical orbit for a satellite, then $R_p + R_a = 2A$, where R_p is the radius $\overline{EP}$ at perigee and R_a is the radius $\overline{ED}$ at apogee; $\overline{CE}$ is called the mean radius and is denoted $\overline{R}$. Then:

$$2\bar{R} = 2A = \overline{PD};$$
$$2\bar{R} = R_p + R_a;$$
$$\bar{R} = \frac{R_p + R_a}{2}. \qquad (1)$$

If 0 is the midpoint of $\overline{EF}$, and $\overline{CO} = d$, then $\overline{EO} = \bar{R} - R_p$, and by the Theorem of Pythagoras,

$$\bar{R}^2 = d^2 + (\bar{R} - R_p)^2$$
$$d^2 = 2R_p\bar{R} - R_p^2$$
$$d^2 = 2R_p\left(\frac{R_a + R_p}{2}\right) - R_p^2$$
$$d^2 = R_pR_a;\ \text{that is, } d = \sqrt{R_pR_a}. \qquad (2)$$

Therefore the equation of the ellipse becomes

$$\frac{X^2}{\bar{R}^2} + \frac{Y^2}{R_pR_a} = 1. \qquad (3)$$

For the Vanguard III satellite, $R_p = 4320$ miles and $R_a = 6330$ miles. Thus, $\bar{R} = (4320 + 6330)/2 = 5325$ miles, and the equation of the elliptical orbit is

$$\frac{X^2}{(5325)^2} + \frac{Y^2}{(4320)(6330)} = 1.$$

As was stated in Chapter 5, there are many factors which effect the orbital path of a satellite such as lunar and solar perturbations, Earth's oblateness, atmospheric density, Earth's pear shape, and atmospheric drag. Although these factors do have some effect and must be accounted for when a satellite is in orbit, the graph of equation (3) will give a very close approximation of the actual path of the satellite. In an effort to facilitate the mathematics in this chapter, the path of a satellite will be determined by an equation such as (3). With this assumption the mathematics will be simplified, but students should be aware that in actual practice, the mathematical theory and calculations are much more complex.

It is often desirable to know the length or circumference of the orbital arc. If the equation of the ellipse is known, there are many formulas that will give an approximate value for the circumference. Some of these equations are:

$C = \pi(a + b)$, where a is the length $\overline{OD}$ of the semimajor axis and b is the length $\overline{OC}$ of the semiminor axis.

$$C = \pi\,[3/2\ (a + b) - \sqrt{ab}]$$

$$C = \pi\sqrt{2(a^2 + b^2)} \qquad (4)$$

The second of the three formulas is best to use when there is a large difference in the lengths of the two axes. The result of the first formula will be too small and that of the third too large, but the average of these two will be very nearly correct.

The circumference of an ellipse can be determined to any degree of accuracy from formula (5) which is derived from calculus.

$$C = 2\pi a\left[1 - \left(\frac{1}{2}\right)^2 \times \frac{e^2}{1} - \left(\frac{1 \times 3}{2 \times 4}\right)^2 \times \frac{e^4}{3} - \left(\frac{1 \times 3 \times 5}{2 \times 4 \times 6}\right)^2 \times \frac{e^6}{5} - \dots\right]$$

or

$$C = 2\pi a\left[1 - \frac{e^2}{4} - \frac{3e^4}{64} - \frac{5e^6}{256} - \frac{175e^8}{16384} - \dots\right] \text{ where eccentricity } e = \frac{1}{a}\sqrt{a^2 - b^2} \quad (5)$$

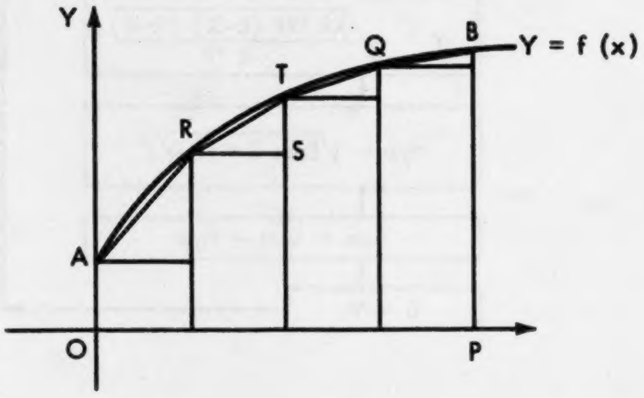


Figure 6–18

The computer can use any formula for circumference. The computer can also be used to find the approximate length of an arc of a curve by using a process similar to that of finding the area under a curve. For example, if the length of arc AB is to be determined in Figure 6–18, a perpendicular is drawn from point B to the X-axis at P. The line segment OP is then divided into N equal increments. If N = 4, as in Figure 6–18, then the lengths of the four chords ($\overline{AR}$, $\overline{RT}$, $\overline{TQ}$, $\overline{QB}$) which subtend the arc can be found. For each triangle (such as ΔRST) the base and altitude are known, and by the Theorem of Pythagoras the hypotenuse can be determined. For example, in ΔRST we have $\overline{RS} = C/4$ and $\overline{ST} = f(C/2) - f(C/4)$ where C is the X-coordinate at P. Therefore,

$$\overline{RT}^2 = (C/4)^2 + [f(C/2) - f(C/4)]^2$$

To calculate the circumference of an ellipse, the length of the arc in the first quadrant can be found and four times this length will give the circumference. For the ellipse in Figure 6–17, one of the X-intercepts is located at $(\overline{R},0)$. From equation (2) a Y-intercept is located at $(0, \sqrt{R_pR_a})$. If the line segment $\overline{OD}$ is divided into N equal intervals, the ordinate at the endpoint of each interval can be calculated by equation (6), which is another form of equation (3).

$$Y = \sqrt{\frac{R_pR_a(\overline{R} - X)(\overline{R} + X)}{\overline{R}^2}} \qquad (6)$$

Note that the numerator of the radicand contains the factors $(\overline{R} - X)$ and $(\overline{R} + X)$ rather than $\overline{R}^2 - X^2$. This was done because the computer can perform additions and subtractions more rapidly than multiplications and divisions. This should be kept in mind when programing. A flow chart for finding the circumference of the ellipse is given in Figure 6–19.

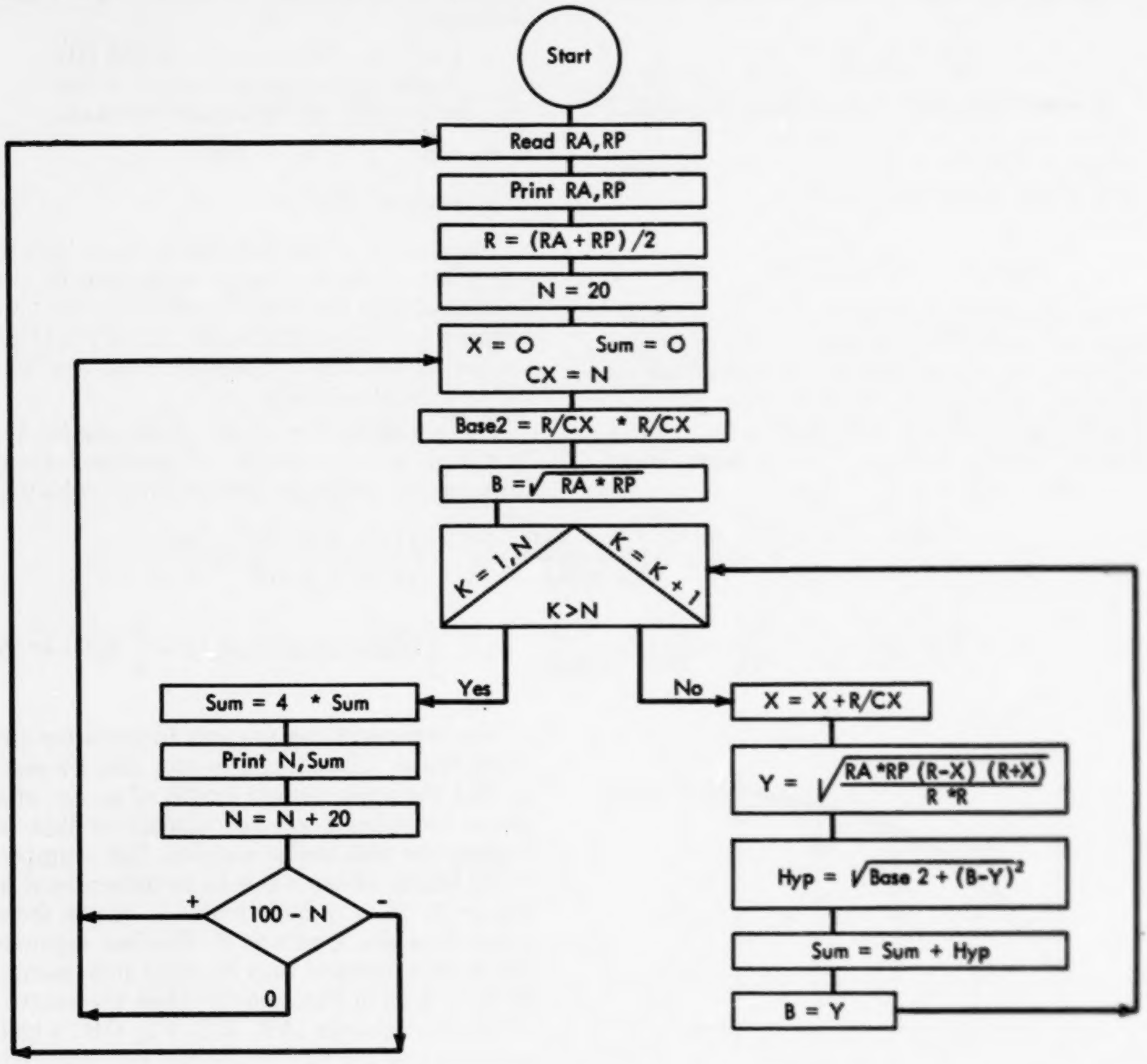

Start
Read RA, RP
Print RA, RP
R = (RA + RP) / 2
N = 20
X = O Sum = O
CX = N
Base2 = R/CX * R/CX
B = √ RA * RP
K = 1, N
K = K + 1
K>N
Yes
No
Sum = 4 * Sum
Print N, Sum
N = N + 20
100 - N
+
-
0
X = X + R/CX
Y = √ (RA * RP (R-X) (R+X)) / (R * R)
Hyp = √ Base 2 + (B-Y)²
Sum = Sum + Hyp
B = Y

Figure 6–19

A Fortran program for finding the circumference is given in Figure 6–20. Note that the data given in the program are the radii at apogee and perigee in statute miles for the satellites Relay, Explorer VIII, Echo I, Explorer VIII, OSO, and Alouette in that order.

```
C     CIRCUMFERENCE OF ELLIPSE GIVEN RADII AT APOGEE AND PERIGEE
    1 READ,RA,RP
      PRINT 5,RA,RP
    5 FORMAT(/2F8.0)
      R=(RA+RP)/2.0
      N=20
    2 X=0.0
      CX=N
      SUM=0.0
      BASE2=R/CX*R/CX
      B=SQRT(RA*RP)
      DO 3 K=1,N
      X=X+R/CX
      Y=SQRT(RA*RP*(R-X)*(R+X)/(R*R))
      HYP=SQRT(BASE2+(B-Y)*(B-Y))
      B=Y
    3 SUM=SUM+HYP
      SUM=4.0*SUM
      PRINT 4,N,SUM
    4 FORMAT(/I6,F10.0)
      N=N+20
      IF (100-N)1,2,2
      END
7000. 4900
4680. 4342
5048. 4948
5423. 4258
4370. 4344
4442. 4425
```

Figure 6–20

The computer output is listed for N = 100 along with the output from using five terms of equation (5). The answers are correct to four significant digits.

Circumference for N = 100	Circumference from equation (5)
37088	37092
28330	28333
31399	31402
30300	30303
27373	27375
27853	27856

6–6 Exercises Circumference of an Ellipse

1. Write a program to find the length of the curve of the equation $Y = X^3 - 9X^2 + 24X$ for that part of the curve between $X = A$ and $X = B$, where A and B are given as follows:

A	B
0.0	5.0
1.0	4.0
0.5	8.5
−1.0	5.0

2. Equation (4) was one of three formulas for finding an approximate value for the circumference of an ellipse. As the value of B approaches the value of A, the value of C approaches $2\pi A$, which is the formula for the circumference of a circle. Let B = A and verify this fact. Write a program to find the circumference of an ellipse by use of each of the three formulas, keeping A fixed at 30 and letting B take on the values 2, 4, 6, . . . , 30. Also find the circumference which is the average of the circumferences of the first and third equations. Finally have the program calculate the circumference by taking the sum of five terms of equation (5). When the computer output is obtained, observe the high degree of accuracy in the results of these formulas.

3. In the previous exercise the circumference of an ellipse was found five different ways. Using the same five formulas and the data given in Figure 6–20, write a program to find the circumference of the orbital paths of the satellites.

6–7 Velocity Along an Elliptical Arc

In effect Kepler's second law states that the line joining the center of the earth to a

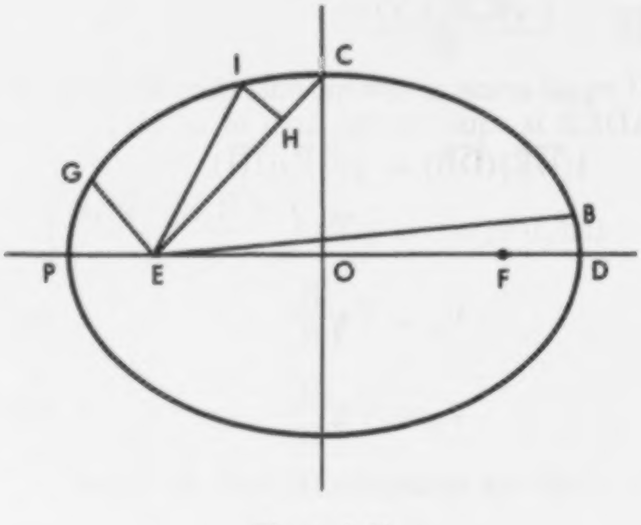


Figure 6–21

satellite sweeps out equal areas in equal intervals of time. By use of this law, consideration can be given to the velocity of a satellite in orbit.

In Figure 6–21 let the velocity at the point D (apogee) be denoted V_a and be called the velocity at apogee. The velocity at perigee is denoted V_p, and the velocity at point C is denoted $\overline{V}$. For a given time interval the satellite moves from D to B, or from C to I, or G to P. For a very small time interval, arcs DB, CI, and GP can be considered as line segments and the sectors DEB, CEI, and GEP of the ellipse can be considered as triangular regions. In ΔDEB, if $\overline{DE}$ is the base, then $\overline{DB}$ will be an approximation for the altitude. The length of $\overline{DB}$ will be $\overline{DB} = (V_a)(t)$, where t denotes a small period of time. Similarly, $\overline{GP} = (V_p)(t)$.

By Kepler's second law, equal areas are swept out. Thus, the area of ΔDEB is equal to the area of ΔGEP ;

$$\tfrac{1}{2}(\overline{ED})(\overline{DB}) = \tfrac{1}{2}(\overline{EP})(\overline{GP})$$

$$\tfrac{1}{2}(R_a)(V_a)(t) = \tfrac{1}{2}(R_p)(V_p)(t)$$

$$R_a(V_a) = R_p(V_p) \qquad (7)$$

In ΔCEI, CI can be considered parallel to PD for a very small period of time, then ∠ICH ≅ ∠CEO. If the altitude $\overline{IH}$ is drawn to base $\overline{EC}$, then two of the angles of ΔCIH equal two of the angles of ΔECO and ΔCIH ∼ ΔECO. Since the corresponding sides will be proportional,

$$\frac{\overline{CO}}{\overline{EC}} = \frac{\overline{IH}}{\overline{CI}} \quad \text{and} \quad \overline{IH} = \frac{(\overline{CO})(\overline{CI})}{\overline{EC}}$$

Since $\overline{CO}$ was given to be equal to $\sqrt{R_aR_p}$ in equation (2),

$$\overline{IH} = \frac{(\sqrt{R_aR_p})(\overline{V})(t)}{\overline{R}}$$

If equal areas are swept out, then the area of ΔDEB is equal to the area of ΔCID;

$$\tfrac{1}{2}(\overline{DE})(\overline{DB}) = \tfrac{1}{2}(\overline{CE})(\overline{IH})$$

$$\tfrac{1}{2}(R_a)(V_a)(t) = \tfrac{1}{2}(\overline{R})\left(\frac{\sqrt{R_aR_p}\,(\overline{V})(t)}{\overline{R}}\right)$$

$$V_a = \overline{V}\sqrt{\frac{R_p}{R_a}} \qquad (8)$$

$$V_p = \overline{V}\sqrt{\frac{R_a}{R_p}} \qquad (9)$$

Multiplying equations (8) and (9) gives

$$\overline{V} = \sqrt{V_aV_p}. \qquad (10)$$

Thus $\overline{V}$ turns out to be the geometrical mean between V_a and V_p and will be called the mean velocity. Notice that the mean velocity occurs at point C.

The following formulas were developed in Chapter 5:

$$F = \frac{GMm}{R^2}$$

$$F = ma$$

$$a = \frac{V^2}{R}$$

From these equations we have

$$V = \sqrt{\frac{GM}{R}} \qquad (11)$$

where the constant $\sqrt{GM} = 1.115 \times 10^6$, R is in miles, and V is in miles per hour. Equation (11) holds for circular orbits and equation (12) gives a very good approximation for elliptical orbits.

$$\overline{V} = \sqrt{\frac{GM}{\overline{R}}} \qquad (12)$$

The period of time for a circular orbit is

$$T = 2\pi R/V$$

Again for an elliptical orbit, a very good approximation is obtained by using equation (13)

$$T = 2\pi\overline{R}/\overline{V} \qquad (13)$$

From equations (12) and (13)

$$T^2 = \frac{4\pi}{GM}(\overline{R})^3$$

$$T \approx 3.38 \times 10^{-4}(\overline{R})^{3/2} \text{ minutes} \qquad (14)$$

Note that this is a form of Kepler's third law which states that the square of the time of a satellite varies directly as the cube of the radius.

To illustrate the use of some of the formulas, let us determine the mean velocity for the Mercury satellite which has a mean radius of 4100 miles. Using equation (12),

$$\overline{V} = \sqrt{GM/4100} \approx 1{,}115{,}000/64.03$$
$$\approx 17400 \text{ mi/hr.}$$

The Syncom satellite has an orbital altitude of 22,300 miles. Therefore $\overline{R} \approx 26{,}300$ miles. By a process similar to that just completed, $\overline{V} \approx 6900$ miles per hour. The circumference of its orbit is $2\pi\overline{R} \approx 52{,}600\,\pi$ miles. The period of time for Syncom to orbit Earth is $T \approx 52600\,\pi/6900$;

$$T \approx 24 \text{ hours.}$$

This gives an explanation of why Syncom seems to be stationary at a certain point above the earth.

6–7 Exercises Velocity Along an Elliptical Arc

1. For Nimbus I, perigee was 260 miles and apogee 580 miles. If the mean velocity for Nimbus was 16,800 miles per hour, find the velocity at perigee and the velocity at apogee.
2. For Explorer XX perigee was 540 miles and apogee was 634 miles. Find the period of time in minutes for one orbit.
3. For Explorer XIX the velocity at perigee was 18,050 miles per hour and the velocity at apogee was 14,200 miles per hour. Find the mean velocity, mean radius, and the period of time in minutes for one orbit.
4. Using the definition of eccentricity that was given in Section 2–4, write a Gotran program that will find the eccentricity of the elliptical orbits for the satellites listed in Figure 6–20.
5. If apogee and perigee are given in statute miles for the following satellites, then write a program that will print R_a, R_p, V_a, V_p, and T for each satellite.

Satellite	Apogee	Perigee
Syncom I	21,650	21,400
Tiros III	510	457
Tiros VIII	468	436
Syncom II	22,900	22,200
Explorer XX	634	540
OSO	358	346

6. By use of equations (1), (7), (10), and (12), derive the formulas

$$R_a = \frac{R_p^2V_p^2}{2GM - V_p^2R_p};$$

$$V_a = \frac{2GM - V_p^2R_p}{R_pV_p}.$$

If perigee and the velocity at perigee are given in statute miles for each of the following satellites, then write a program that will print R_a, R_p, V_a, V_p, and T.

Satellite	Perigee	Velocity at perigee
Syncom III	22,160	6,870
Echo II	642	16,540
Alouette	425	16,790
Gemini	160	17,290

6–8 Curve Fitting

The problem of determining the equation from a set of given data is common in mathematics and science. For example, one might want to find the equation of the elliptical orbit of a satellite if the coordinates of several points along its path were known. This process of fitting a curve to given points often yields very useful results.

We first consider linear functions of the form

$$y = mx + b$$

Quadratic functions will be considered later in this section.

If the coordinates of two points of a line are known, then the equation of the line can be found. If the given points are (X_1, Y_1) and (X_2, Y_2), then

$$Y_1 = mX_1 + b,$$
$$Y_2 = mX_2 + b;$$
$$m = \frac{Y_1 - Y_2}{X_1 - X_2}; \qquad (15)$$
$$b = Y_1 - mX_1 = Y_1 - \left(\frac{Y_1 - Y_2}{X_1 - X_2}\right)X_1$$
$$= \frac{X_1Y_2 - X_2Y_1}{X_1 - X_2} \qquad (16)$$

For the points (0, 1) and (1, 3), $m = 2$, $b = 1$, and the linear equation becomes

$$Y = 2X + 1$$

Problems often arise in which there is reason to believe that a linear function may be used to describe or approximate a given situation. If the coordinates of many points are given, then various methods can be employed to find a linear equation that provides a "good fit" to the data. One method is called the *method of average points*. To illustrate this method consider the data

$$(X_1, Y_1), (X_2, Y_2), (X_3, Y_3), (X_4, Y_4), (X_5, Y_5), (X_6, Y_6).$$

In general, the set of data is divided into two subsets of approximately the same number of elements. We shall consider the subset

$$(X_1, Y_1), (X_2, Y_2), (X_3, Y_3)$$

and the subset

$$(X_4, Y_4), (X_5, Y_5), (X_6, Y_6).$$

Then we find the average of the X-coordinates, and the average of the Y-coordinates for each subset. Suppose these average values are X_a and Y_a for one set; X_b and Y_b for the other set;

$$X_a = (X_1 + X_2 + X_3)/3$$
$$Y_a = (Y_1 + Y_2 + Y_3)/3$$
$$X_b = (X_4 + X_5 + X_6)/3$$
$$Y_b = (Y_4 + Y_5 + Y_6)/3$$

Finally the equation of the line joining (X_a, Y_a) and (X_b, Y_b) may be found by use of equations (15) and (16).

The equation obtained by the method of average points will not yield a unique equation since the manner in which the given data is grouped will effect the result obtained. This is illustrated for the data $(-2, 11)$, $(-1, 8)$, $(2, 5)$, $(1, 2)$, $(3, 1)$, and $(4, -2)$. If the first three sets of ordered pairs are grouped and the last three are grouped, then $X_a = -1/3$, $Y_a = 8$, $X_b = 8/3$, $Y_b = 1/3$ and the equation is $Y = -\frac{23X}{9} + \frac{193}{27}$. If $(-2, 11)$, $(2, 5)$, and $(1, 2)$ are grouped and the remaining elements are grouped, then the equation becomes $Y = -\frac{11}{5}X + \frac{101}{15}$.

There is another method of curve fitting called the method of least squares. If this method is applied to curve fitting for a linear function, a unique equation will result. Furthermore, the equation will fit the given data about as well as possible. As might be expected, the method of least squares involves much more work.

To develop the method of least squares for a linear function, consider the equation $Y = mX + b$, which may be written

$$(mX + b) - Y = 0. \qquad (17)$$

If a point lies on this line and the coordinates of the point are substituted into the left member of equation (17), then the sum of the terms will equal zero. If the coordinates of a point, which is slightly off the line, are substituted into the left member of equation (17), the sum of the terms will not equal zero. This can be represented by the equation

$$R = (mX + b) - Y \qquad (18)$$

The magnitude of R provides a measure for the error that is made in assuming that the point with a given X-coordinate is on the line. Notice that for each point, R is a signed number and thus there could be compensating errors that would not appear when the errors (values of R) were added. All errors show up when we use values of R^2.

The method of least squares involves squaring both members of equation (18) and substituting the coordinates of the given points into this equation. Thus for data

$$(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n),$$

the equations would be

$$R_1^2 = [(mX_1 + b) - Y_1]^2$$
$$R_1^2 = m^2X_1^2 + 2bmX_1 + b^2 - 2mX_1Y_1 + 2bY_1 + Y_1^2$$
$$R_2^2 = m^2X_2^2 + 2bmX_2 + b^2 - 2mX_2Y_2 - 2bY_2 + Y_2^2$$
$$\cdots$$
$$R_n^2 = m^2X_n^2 + 2bmX_n + b^2 - 2mX_nY_n - 2bY_n + Y_n^2$$

The curve that is considered best is obtained when the sum of all the R^2, denoted as ΣR^2, has a minimum value. In algebra it is shown that the quadratic function $AX^2 + BX + C$ with $A > 0$ has a minimum value when

$$X = -B/2A;$$

that is,

$$2AX + B = 0. \qquad (19)$$

This concept can be used in the methods of least squares. When ΣR^2 is found, a quadratic function in m results as shown in equation (20)

$$\Sigma R^2 = m^2\Sigma X^2 + 2bm\,\Sigma X + nb^2 - 2m\Sigma XY - 2b\Sigma Y + \Sigma Y^2 \qquad (20)$$

If equation (20) is a quadratic function in m, then a minimum value occurs since the coefficient of m^2 is positive. Therefore by equation (19) the minimum value occurs when

$$2m\Sigma X^2 + 2b\Sigma X - 2\Sigma XY = 0;$$
$$m\Sigma X^2 + b\Sigma X = \Sigma XY \qquad (21)$$

If equation (20) is considered to be a quadratic function in b, equation (22) is obtained by a similar process:

$$m\Sigma X + nb = \Sigma Y \qquad (22)$$

When equations (21) and (22) are solved for m and b, the desired linear function is obtained.

The process of curve fitting for exponential equations of the form $Y = AB^x$ and for power equations of the form $Y = AX^B$ can always be reduced to that of curve fitting for a straight line by use of special graph paper.

Curve fitting for the parabola will be discussed in one of the exercises. The same process can be used for the ellipse or other types of curves.

6-8 Exercises Curve Fitting

X	−5.2	2.9	−.4	−5.05	1.1	−1.11	−.71	2.38	−2.4	−.7	2.45	−2.4	.15	−4.88
Y	11.33	−10.55	4.1	11.33	−5.2	4.73	2.20	−8.75	7.37	10.35	−6.90	7.5	.30	9.30

1. The ordered pairs that are used for data in this problem are the coordinates that represent the position of stars at a particular instant as taken from a north polar star chart.

Use the method of average points and write a program to determine the coefficients m and b of a linear equation $Y = mX + b$ of a function to fit these points. Make the program flexible enough so that it will read up to thirty ordered pairs of data.

If the data are arranged so that the values of X are increasing or decreasing and the subsets selected by grouping the points in order according to their x-coordinates, then more accurate results will be obtained than if the data are left in random order. Therefore, have the program arrange the ordered pairs in increasing order as determined by X.

2. Using the method of least squares for the data given in Exercise 1 write a program to find the coefficients of the linear function. You may compare this equation with that in the previous exercise by plotting the data on graph paper along with the graphs of the equations found in Exercise 1 and this exercise.

3. Using a technique similar to that used to derive equations (21) and (22), derive equations that will determine the coefficients A, B, and C of the quadratic function $Y = AX^2 + BX + C$. The three formulas that are obtained can be used for curve fitting for the parabola. It may be helpful to follow this procedure:

(a) Square both members of the equation $R = (AX^2 + BX + C) - Y$.

(b) Substitute the data (X_1, Y_1), (X_2, Y_2), . . ., (X_n, Y_n) into the resulting equation.

(c) Find ΣR^2.

(d) The equation corresponding to (20) will be a quadratic function in A. Determine the conditions for which a minimum value of the function occurs. Follow a similar procedure for B and C.

By observing the pattern in the equations just developed, derive a set of equations for a polynomial function of degree three.

4. Use the results of Exercise 3 and write a program to find the coefficients A, B, and C for a quadratic function $Y = AX^2 + BX + C$ to fit the following data:

X	−2.0	−1.4	−1.0	−0.4	−0.2	0.4	1.1	1.6	2.1	2.4
Y	1.9	0.3	0.0	−0.3	−0.1	−0.1	0.5	0.9	1.9	2.3

ANSWERS TO EXERCISES

Chapters 1–6

Chapter 1 —DESCRIBING THE SHAPE OF THINGS

1–1 Shapes on Earth

1. A few examples of plane surfaces are a floor, wall, the cover of a book.
2. One and only one straight line can be drawn through two points.

1–2 Earth's Atmosphere

1. The circles appear to be of the same size; radii of congruent circles are congruent.
2. Concentric circles may be seen on an archery target, other "bullseye" targets, the decorations of many dishes, and in other places.
3. Among the many objects that are approximately spherical are baseballs, golf balls, oranges, and the moon.
4. The radius of a sphere is the distance from the center to any point on the surface of the sphere.

1–3 Angles and Arcs

1. 2 x $\frac{22}{7}$ x 4,000; that is, about 25,000 miles.
2. About 4,000 miles.
3. About 69 miles.

1–4 Positions on Earth

1. In 1 hour, 15°; in 6 hours, 90°; in 12 hours, 180°.
2. 180°.
3. (a) 12,430 miles, (b) 6,215 miles.
4. Yes.
5. No.
6. (a) 21,600 miles, (b) 16,200 miles.

1–5 Observations of Earth

1. About 100,000,000 square miles; that is, in scientific notation about 1.0 x 10^8 square miles.
2. About 4.0 x 10^7 square miles.
3. About 503,000 square miles.

1–6 Maps and Distances

1. Yes, when $\overline{AB}$ is parallel to $\overline{CD}$.
2. Yes, when $\overline{AB} \perp \overline{CD}$.
3. No.
4. A circular region (that is, a circle, and its interior points) with a radius equal to the radius of the sphere.

1–7 Measurements

1. 30 feet.
2. 3 feet.
3. Each object is perpendicular to its shadow. In the case of Exercise 2 this means that the person is standing upon level ground.

Chapter 2 —THE UNIVERSE WE LIVE IN

2–1 Where Do You Live?

1. Think of corners of the room as intersections of walls, the intersections of the ceiling and walls, edges of a desk top as the intersection of the top and sides, and so forth.
2. Think of the points on a wall that are equidistant from the points on the floor at the ends of that wall, and so forth.
3. Think of a coordinate system for the chess board.

2–2 Relative Positions on Earth

1. Henryetta, Oklahoma
2. 96° west longitude; 35° 27.6′ north latitude.

   The following steps may be used to obtain the answers for this exercise:

   (1). Measure the distance between the 32° parallel of latitude and the 36° parallel of latitude. Call this measure A.

   (2). Measure the distance from the 32° parallel of latitude to city in question. Call this measure B.

   (3). On the map used by the author parallels of latitude were 4° apart and the measures were:

   $A = 8.50$ cm; $B = 7.36$ cm.

   (4). Let x = degrees of latitude city in question is above the 32° parallel of latitude.

   (5). $\frac{B}{A} = \frac{7.36}{8.50}$

   (6). $\frac{B}{A} = \frac{x}{4^\circ}$; 4° is span of parallels of latitude on map.

   (7). $\frac{7.36}{8.50} = \frac{x}{4^\circ}$

   (8). $x = 3.46^\circ$ Since $1^\circ = 60'$, $0.46^\circ = 27.6'$

   (9). Latitude of city in question is $32^\circ + 3^\circ\, 27.6'$

   (10). Latitude is 35° 27.6′ north latitude

   (11). Longitude is about 96° west longitude.
3. This exercise may be solved by using the scale stated on the map. Measure the distances on the map; use the scale on the map to determine the numbers of miles represented by the map distances and compare. About 340 miles is saved.
4. No. Use a protractor to observe that the measures of opposite angles are not equal. The angle formed at New Orleans is about 84°; the angle formed at New York is about 76°.

2–3 Fallacies of maps

Trial	Quadrant
F–1	II
K–3	III
C–4	II
K–5	I
M–7	II
K–8	III
C–9	II
M–10	II
K–11	II
C–12	II
M–13	II
K–14	IV
C–15	II
M–16	IV

2–4 The Solar System

Answers to 2–4 are included in the section pages 38 and 39.

2–5 Earth—a Satellite with Satellites

1. $0 + 30 \equiv 6 \pmod{24}$

   $0 + 30 \equiv 30 \pmod{36}$

   Use the chart described in Figure 2–35, align the 0° longitude mark on the third circle with 6 on the hour circle. The position of the satellite will correspond to 30 on the first circle and is about 151° west longitude.
2. 142° west longitude.

2–7 Our galaxy, the Milky Way

1.

Star and Constellation	α	δ
ζ Taurus	5^h36^m	+21°07′
β Orion	5^h13^m	− 8°14′
γ Orion	5^h23^m	+ 6°19′
δ Orion	5^h30^m	− 0°19′
ε Orion	5^h34^m	− 1°13′
ζ Orion	5^h39^m	− 1°58′
κ Orion	5^h46^m	− 9°41′
α Orion	5^h53^m	+ 7°24′
η Gemini	6^h13^m	+22°31′
γ Gemini	6^h36^m	+16°26′
α Gemini	7^h32^m	+31°58′
β Gemini	7^h43^m	+28°07′

2. The desired answers are included in the following completed table.

Nearest Hour Circle	Constellation Position	Star Name	(To nearest minute of time) Right Ascension	(To nearest minute of arc) Declination
1	β Andromeda	Mirach	1^h08^m	+35°26′
2	α Aries	Hamal	2^h05^m	+23°18′
3	β Perseus	Algol	3^h05^m	+40°46′
4	α Taurus	Aldebaran	4^h34^m	+16°26′
5	α Auriga	Capella	5^h14^m	+45°58′
6	α Orion	Betelgeuse	5^h53^m	+ 7°24′
7	α Canis Major	Sirius	6^h44^m	−16°40′
8	β Gemini	Pollux	7^h43^m	+28°07′
9	α Cancer		8^h57^m	+12°00′
10	α Leo (Leonis)	Cor Leonis	10^h07^m	+12°08′
11	β Ursa Major	Merak	11^h00^m	+56°34′
12	α Corvus		12^h07^m	−24°32′
13	α Virgo	Spica	13^h23^m	−10°59′
14	α Boötes		14^h14^m	+19°22′
15	β Boötes	Nekkar	15^h01^m	+40°32′
16	β Scorpius	Acrab	16^h03^m	−19°43′
	α Scorpius	Antares	16^h27^m	−26°21′
17	α Hercules		17^h13^m	+14°26′
18	γ Saggitarius		18^h04^m	−30°26′
19	σ Saggitarius		18^h53^m	−26°21′
20	α Aquila	Altair	19^h49^m	+ 8°46′
21	α Cygnus	Deneb	20^h40^m	−45°09′
22	α Aquarius		22^h04^m	− 0°29′
23	α Pegasus (Pegasi)	Marhab	23^h03^m	+15°01′
24	α Andromeda	Alpheratz	0^h07^m	+28°54′

3. 1 hour and 8 minutes is 15° and 120′; that is, 17°.

ANGLE	SINE	COSINE	TANGENT	ANGLE	SINE	COSINE	TANGENT
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.0145	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5873	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
45°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8093	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14°3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000	90°	1.0000	.0000	

Chapter 3 —MEASUREMENT, A WINDOW TO THE UNIVERSE

Section 3–1—Direct linear measurement

1. The ratios should be the same, in the two systems.
2. The following is a set of sample data for measurements in inches;

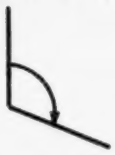
3.48	3.43	3.45	3.43
3.44	3.44	3.41	
3.50	3.41	3.42	

The average value for the length of the line is 3.44 inches.

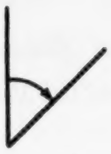
Section 3–2—Direct Angular Measurement

1. (a) 26°; (b) 134°; (c) 15°; (d) 340°

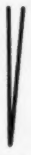
2.



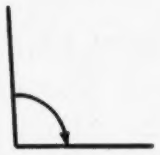
112.5°
(a)



47°
(b)

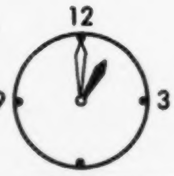


210'
(c)

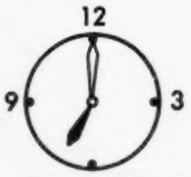


360,000" (100°)
(d)

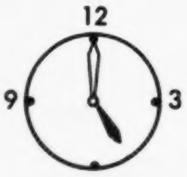
3.



1 1/2 HOURS
45°
(a)



7 1/3 HOURS
220°
(b)



5 HOURS
150°
(c)

4.

	Degrees	Minutes	Seconds
(a)	15°	900′	54,000″
(b)	40°	2,400′	144,000″
(c)	0.083°	5′	300″
(d)	0.0083°	0.5′	30″

Section 3–3—Indirect Measurement

About 1.1 miles

Section 3–4—Measurement of Earth

About 2,990 miles

Section 3–6—Earth as Viewed from a Satellite

1.

	Altitude of the satellite (miles)	Visible Surface (square miles)	Part of Earth's surface visible
(a)	200	4,787,000	0.02%
(b)	1000	20,096,000	10%
(c)	4000	50,265,000	25%
(d)	5000	55,851,000	28%

2. 4
3. $\text{Cos } \angle e = 0.8000; \angle e \approx 36.9; \overline{GC}$

$$\approx \frac{73.8}{360} \times 2\pi \times 4000 \approx 5{,}150 \text{ miles}$$

3–7 Distance to the Moon

1. 1,500,300 miles.
2. 2,270 miles per hour.
3. The moon is about 253,200 miles at maximum distance and about 222,300 miles at minimum distance.

3–8 The Yardstick of Space

1. About 583,412,000 miles; about 66,600 miles per hour.
2. 31.5′.
3. 32.5′.
4. The sun's diameter is about 108 times the diameter of the Earth.
5. Over 1,000,000.

3–9 The Inner Planets

1. 0.25 A.U.
2. 1.75 A.U.
3. 0.625 A.U.
   1.375 A.U.
4. 1.6

Section 3–5—Altitude of a Model Rocket

Observer A				Observer B				
d (ft)	Scale H	Scale V	Altitude a	Scale H	Scale V	Altitude $\overline{DC}$	Average Altitude	Acceptable
1000	30°	30°	500	60	45	500	500	yes
1000	40°	40°	787	65	50	793	790	yes
1000	37°	48°	1066	68	60	1079	1072	yes
1000	52°	31°	570	70	32	486	528	no
1000	12°	56°	1601	104	81	1461	1531	yes
1000	64°	73°	2333	43	68	2326	2320	yes

3–11 Distances to the Stars

	Parallax	Parsecs	Light year	A.U.
1.	0.4″	2.5	8.15	513,450
2.	0.018″	55	179.3	11,295,900
3.	0.87″	1.15	3.74	235,620
4.	0.049″	20.6	67	4,221,000
5.	0.022″	45	146.7	9,242,100

3–12 Magnitude and Brightness

1. 9.5
2. 2.5
3. The sun is about 420,000 times brighter than the moon.
4. −2.1
5. 1,120

3–13 Apparent and Absolute Magnitude

1. (a)

Star	Distance (parsecs)	Apparent magnitude	Absolute magnitude
A	10	2	2
B	100	4	−1
C	40	0	−3

(b) C,A,B

(c) C,B,A

2.

	M	m	Parsecs	Distance light years	Parallax
(a)	0.4	−1.8	3.57	11.6	0.28″
(b)	−2.8	−0.3	33.3	108.5	0.03″
(c)	7.9	7.9	10	32.6	0.1″
(d)	4.37	1.2	2.2	7.17	0.45″

3–14 Classification of Stars

1. About 2.1; 0.8 magnitudes
2. About 40,000
3. About 4.1 times greater
4. About 14 light years

Chapter 4 —MOTION IN SPACE

4–1 What Is Motion?

In Figures 4–2 and 4–3 the coordinate planes are at right angles to each other. To find the length of $\overline{AB}$, first find the length of $\overline{ED}$ in the right triangle ECD where $\overline{EC} = 60$, $\overline{CD} = 10$, and $\angle ECD = 90°$;

$$ED = \sqrt{60^2 + 10^2} \approx 10\sqrt{37}$$

In the plane of $\overline{ED}$ and $\overline{AB}$ construct a line $\overline{AD'}$ through A parallel to $\overline{ED}$ such that $\overline{ED} = \overline{AD'}$, and $\overline{AD'}$ is perpendicular to $\overline{BD}$. In the right triangle AD′B,

$\overline{AD'} = 10\sqrt{37}$

$\overline{BD'} = 30$

$\overline{AB} = \sqrt{(10\sqrt{37})^2 + 30} \approx 10\sqrt{46} \approx 67.8.$

The change of rate column of Table 4–2 may be explained as follows:

The v's from top to bottom are 0, 25, 50, 50, 50, 50, 50, 50, 50, 50, 50. These are differences in average speeds. The time interval is one second with the exception of the first difference which is a half second (from 0 time to the midpoint of the first second interval). The acceleration is constant, 50 ft/sec².

1. $r = \dfrac{d}{\Delta t}$

$\Delta t = \dfrac{d}{r}$

$\Delta t = \dfrac{135{,}000{,}000 \text{ mi}}{186{,}000 \text{ mi/sec}}$

$= \dfrac{1.35 \times 10^8 \text{ mi}}{1.86 \times 10^5 \text{ mi/sec}}$

$\Delta t \approx 7.25 \times 10^2 \text{ sec} = 725 \text{ sec.}$

2. $r = 4000 \text{ mi} + 1000 \text{ mi} = 5000 \text{ mi}$

$d = 2\pi r$

$= 2\pi 5000 \text{ mi}$

$r = \dfrac{3.14 \times 10^4 \text{ mi}}{1.183 \times 10^2 \text{ min}}$

$\approx 265 \text{ mi/min}$

3. $r = 4000 \text{ mi} + 630 \text{ mi} = 4630 \text{ mi}$

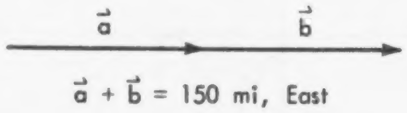
$d = 2\pi 4630 \text{ mi} = 2.92 \times 10^4 \text{ mi}$

$r = \dfrac{2.92 \times 10^4 \text{ mi}}{1.054 \times 10^2 \text{ min}} \approx 277 \dfrac{\text{mi}}{\text{min}}$

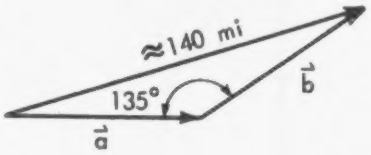
From the problem there are two known differences in the orbits: (1) altitude, and (2) time for one revolution. Guesses (hypotheses) might be conjectured as to how these are related. The weight of Echo I was not stated. Does weight make a difference in speed? The answers are rather subtle. This topic is considered in Chapter 5.

Section 4–2 Road Maps Without Roads

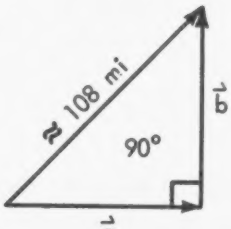
1.



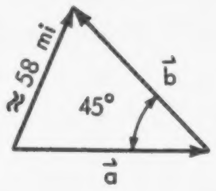
2.



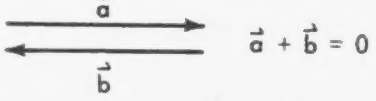
3.



4.



5.



6. The magnitude of $\vec{a} + \vec{b}$ is greater than or equal to zero and less than or equal to the sum of the magnitudes of the given vectors.

Section 4–3 Velocity Vectors

1. $\vec{v}_a = 2000 \text{ ft/sec}$

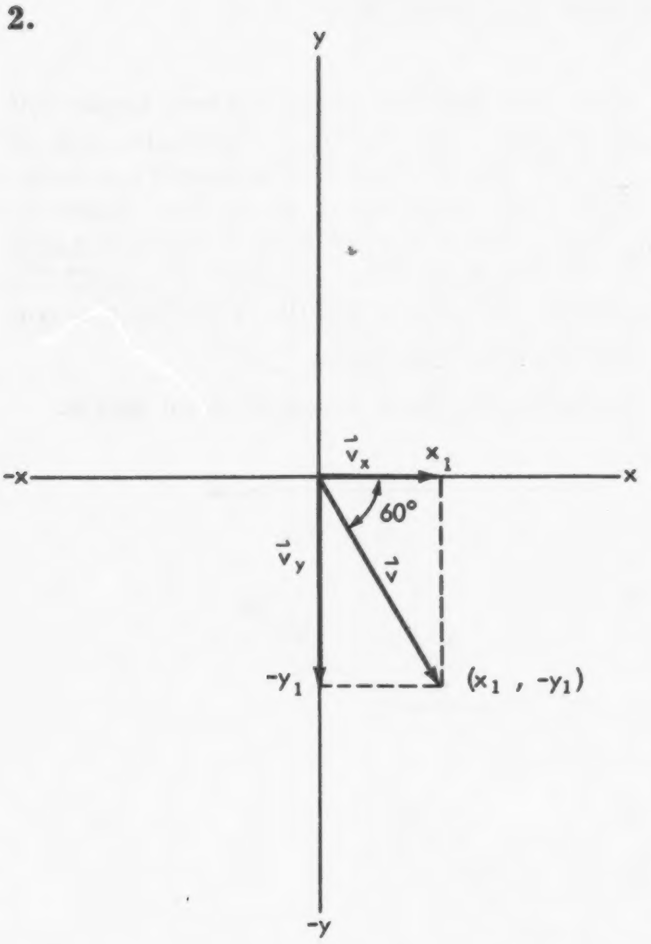
$\vec{v}_h = 200 \text{ ft/sec}$

$\vec{v}_t = 2010 \text{ ft/sec}$

$\tan\theta = \dfrac{200}{2000} = 0.100$

$\angle\theta = 5.7°$

2.



The figure presents the problem in vector representation; $\vec{v}$ terminates at $(x_1, -y_1)$. The projection of $\vec{v}$ on the x-axis is $\vec{v}_x$ and, likewise, $\vec{v}_y$ is the projection of $\vec{v}$ on the y-axis. Notice that 30°–60° right triangles can be formed so that

$$v_x = \frac{v}{2} = \frac{3 \times 10^3 \text{ ft/sec}}{2}$$
$$= 1.5 \times 10^3 \text{ ft/sec}$$
$$v_y = \sqrt{3}v_x = (1.73)(1.15 \times 10^3 \text{ ft/sec})$$
$$= 2.59 \times 10^3 \text{ ft/sec}$$

Section 4–4 Acceleration Vectors

1. (a) Depress the gas pedal to gain speed.
   (b) Apply the brakes to reduce the speed.
   (c) Turn the steering wheel to change direction.

2. $v_t = a\Delta t$

$$v_5 = (100 \text{ ft/sec}^2)(5 \text{ sec})$$
$$= 500 \text{ ft/sec}$$
$$v_{10} = (100 \text{ ft/sec}^2)(10 \text{ sec})$$
$$= 1000 \text{ ft/sec}$$
$$\bar{v}_5 = \frac{0 + 500 \text{ ft/sec}}{2} = 250 \text{ ft/sec}$$
$$d = \bar{v}t$$
$$= (250 \text{ ft/sec})(5 \text{ sec}) = 1250 \text{ ft}$$
$$\bar{v}_{10} = \frac{0 + 1000 \text{ ft/sec}}{2} = 500 \text{ ft/sec}$$
$$d = (500 \text{ ft/sec})(10 \text{ sec}) = 5000 \text{ ft}$$

At $t = 5$, $v_5 = 500$ ft/sec and $d_5 = 1250$ ft. At $t = 10$, $v_{10} = 1000$ ft/sec and $d_{10} =$ 5000 ft.

When t was doubled, v was doubled but d was 4 times as great. This is characteristic of accelerated motion. The relationship is (for accelerated motion):

$$v \propto t$$
$$a \propto t^2$$

Section 4–6 Analysis of Projectile Motion

For the half second intervals (½, 1½, 2½, etc.), the coordinates are: (+50, −4); (+150, −36); (+250, −125); (+350, −196); (+450, −324).

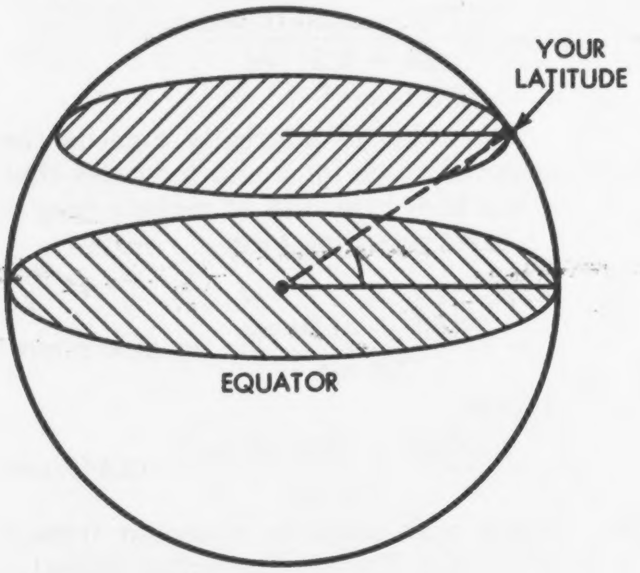
The horizontal velocity and the time in flight which is determined by the maximum height of the object determine the horizontal distance (range).

In as much as Earth is not flat, the point of impact would have a −y value greater than the initial height, hence the object would fall longer and further. If the range were great these considerations would have to be considered. At great height and great horizontal velocities it is conceivable that the object will not hit the Earth. It will orbit!

Section 4–7 Circular Motion

$$a = \frac{4\pi^2 r}{T^2} \qquad (1)$$
$$v = \frac{2\pi r}{T} \qquad (2)$$
$$T = \frac{2\pi r}{v} \qquad \text{from (2)}$$
$$T^2 = \frac{4\pi^2 r^2}{v} \qquad (3)$$
$$a = \frac{4\pi^2 r}{4\pi^2 r^2/v^2} \qquad \text{substitute (3) in (1)}$$
$$a = \frac{v^2}{r}$$

Section 4–8 Angular Velocity



The angular velocity of all locations on Earth are equivalent. The speeds will vary with the maximum at the Equator (a great circle), since all other locations circumscribe a circle of smaller radius in equivalent time periods.

Launchings at Cape Kennedy take advantage of the West to East velocity of an object on Earth and the greater speed near the equator.

Section 4–10 Mass

The space vehicle with the smaller change in velocity has twice as much mass.

Section 4–11 Units of Measure

1. wt = Mg, and g is the same for all objects at the same location. Therefore if

$$wt_1 = wt_2$$
$$M_1g = M_2g$$
$$M_1 = M_2$$

2. $F\Delta t = mv$

$$\Delta v = F\Delta t/m$$
$$= \frac{(10 \text{ kg-m/sec}^2)(5 \text{ sec})}{10 \text{ kg}}$$
$$= 5 \text{ m/sec}$$
$$a = \frac{5 \text{ m/sec}}{5 \text{ sec}}$$
$$= 1 \text{ m/sec}^2$$

In this case we have kept the units as a physicist would do.

Section 4–13 Rocket Engines

800 ft/sec. Yes; a 10,000 pound thrust for 25 seconds will produce the same change as a 50,000 pound thrust for 5 seconds on the same rocket.

Section 4–14 Sounding Rockets

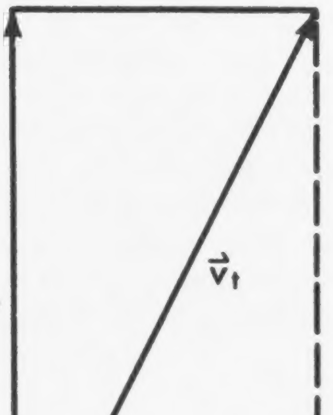
1. 519,000 ft; (See Figure 4–26)

$$\overline{v}_y = \frac{519{,}000 \text{ ft.}}{198.5 \text{ sec}} = \frac{5.19 \times 10^5 \text{ ft}}{1.985 \times 10^2 \text{ sec}}$$
$$= 2.68 \times 10^3 \text{ ft/sec}$$

2. 270,000 ft; (See Figure 4–27)

$$\overline{v}_x = \frac{2.7 \times 10^5 \text{ ft}}{1.985 \times 10^2 \text{ sec}}$$
$$= 1.36 \times 10^3 \text{ ft/sec}$$

3.



$$\overline{v}_t = \sqrt{(2.68 \times 10^3)^2 + (1.36 \times 10^3)^2}$$
$$\overline{v}_t = 3 \times 10^3 \text{ ft/sec}$$

Direction

$$\tan\theta = 2.68/1.36 = 1.97$$
$$\angle\theta = 63.2^\circ$$

4. 540,000 ft. (See Figure 4–27)

5. $\frac{5.4 \times 10^5}{3.85 \times 10^2} = 1.4 \times 10^3 \text{ ft/sec}$

6. Vertical velocity ($\overline{v}_y$) (See Figure 4–26)

$$\overline{v}_y = \frac{(371{,}000 - 305{,}000) \text{ ft}}{20 \text{ sec}}$$
$$= 3300 \text{ ft/sec}$$

Horizontal velocity ($\overline{v}_x$) (See Figure 4-27)

$$\overline{v}_x = \frac{(127{,}000 - 97{,}000)\text{ ft}}{20\text{ sec}}$$

$$= 1500\text{ ft/sec}$$

True velocity ($\overline{v}_t$)

$$\overline{v}_t = \sqrt{(3300)^2 + (1500)^2} = 3620\text{ ft/sec}$$

$$\tan\theta = \frac{3300}{1500} = 2.2$$

$$\angle\theta = 65.5^\circ$$

7. Phase 1: is the initial steep slope which indicates high positive acceleration.

   Phase 2: The slope of the line goes down as drag and gravity retard the velocity.

   Phase 3: A steep positive slope for about 6.4 seconds is due to the thrust by the 2nd stage motor.

   Phase 4: Starts with the burnout of the 2nd stage and ends with apogee. In this interval the rocket is rising but losing vertical velocity.

   Phase 5: From apogee to impact. High gain in velocity due to gravity.

8. The forces of gravity and drag produce an acceleration in the direction opposite to the velocity of the rocket.

   $$a = \frac{(1770 - 3150\text{ ft/sec}}{(20 - 3.5)\text{ sec}}$$

   $$\approx -83.6\text{ ft/sec}^2$$

   This value of a greatly exceeds the accepted value for g and indicates that at the high velocities of rockets drag is a serious retarding force.

9. *Nike*

   $$\bar{a} = \frac{(3150 - 0)\text{ ft/sec}}{3.5\text{ sec}} = 90.0\text{ ft/sec}^2$$

   Apache

   $$\bar{a} = \frac{(5700 - 1770)\text{ ft/sec}}{6.4\text{ sec}} = 614\text{ ft/sec}^2$$

10. Flight path angle is measured from a horizontal. The positive value indicates that angle is measured above the horizontal (rocket gaining altitude) and the negative value indicates that the angle is measured below the horizontal (rocket losing altitude).

11. Figure 4-28 gives directional data and Figure 4-29 gives magnitude data. By using the method called for in Exercise 6, you could use Figures 4-26 and 4-27.

Chapter 5 —SPACE MECHANICS

Section 5–1

1. 0.04 seconds.
2. $s = 299t^2$
3. Approximately 84 cm/sec.
4. Approximately 608 cm/sec^2.

Section 5–2

1. Approximately 24.75 cm.
   Displacement from Table 5–1 is 24.72 cm.
2. (a) 6.6 seconds.
   (b) 363 feet.
3. (a) 16 feet, 144 feet, and 400 feet.
   (b) 32 ft/sec., 96 ft/sec., and 160 ft/sec.
4. (a) 3 seconds, (b) 144 feet, (c) 1 and 5 seconds after projection.

Section 5–3

1. (a) A construction, (b) A construction, (c) 11.32 newtons, (d) 1.68 m/sec.
2. 2.8 m/sec.

Section 5–4

1. 618 miles.
2. 8 ft/sec^2
3. 5.97×10^{24} kilograms (6.57×10^{21} tons).
4. 24,100 miles from the moon or 215,900 miles from the earth.
5. A derivation.
6. (a) 7,640 m/sec. (25,059 ft/sec. or 17,083 mi/hr).
   (b) 1,500 m/sec. (4.920 ft/sec. or 3,354 mi/hr).
7. (a) A graph, (b) approximately 12.2 ft/sec^2, (c) Mars.

Section 5–5

1. (a) 9.380×10^7 meters (58,260 miles).
   (b) 231 m/sec. (516.5 mi/hr).
2. 6845 seconds (114.08 minutes).
3. A derivation.

Section 5–6

1. 7.044×10^8 joules (5.283×10^8 ft (lbs))
2. (a) Approaches 1 (b) $\sqrt{\dfrac{2GM}{R_1}}$.
3. (a) 10,800 m/sec (35,424 ft/sec or 24,149 mi/hr).
   (b) 10,200 m/sec (33,456 ft/sec or 22,807 mi/hr).
4. 3,002 m/sec (9,847 ft/sec or 6,712 mi/hr).

Section 5–7

1. A derivation.
2. $E = GmM\left(\dfrac{1}{r} - \dfrac{1}{2a}\right)$. Both have the same energy.
3. A satellite has the same angular momentum at apogee and perigee. Therefore, it appears that angular momentum is conserved.
4. (a) $V_p = 7{,}420$ m/sec (24,338 ft/sec or 16,591 mi/hr)
   $V_a = 7{,}150$ m/sec (23,452 ft/sec or 15,987 mi/hr)
   (b) $e = 0.019$
   (c) 6,510 seconds (108.5 minutes)

Detailed solutions may be found on pages 146 through 158.

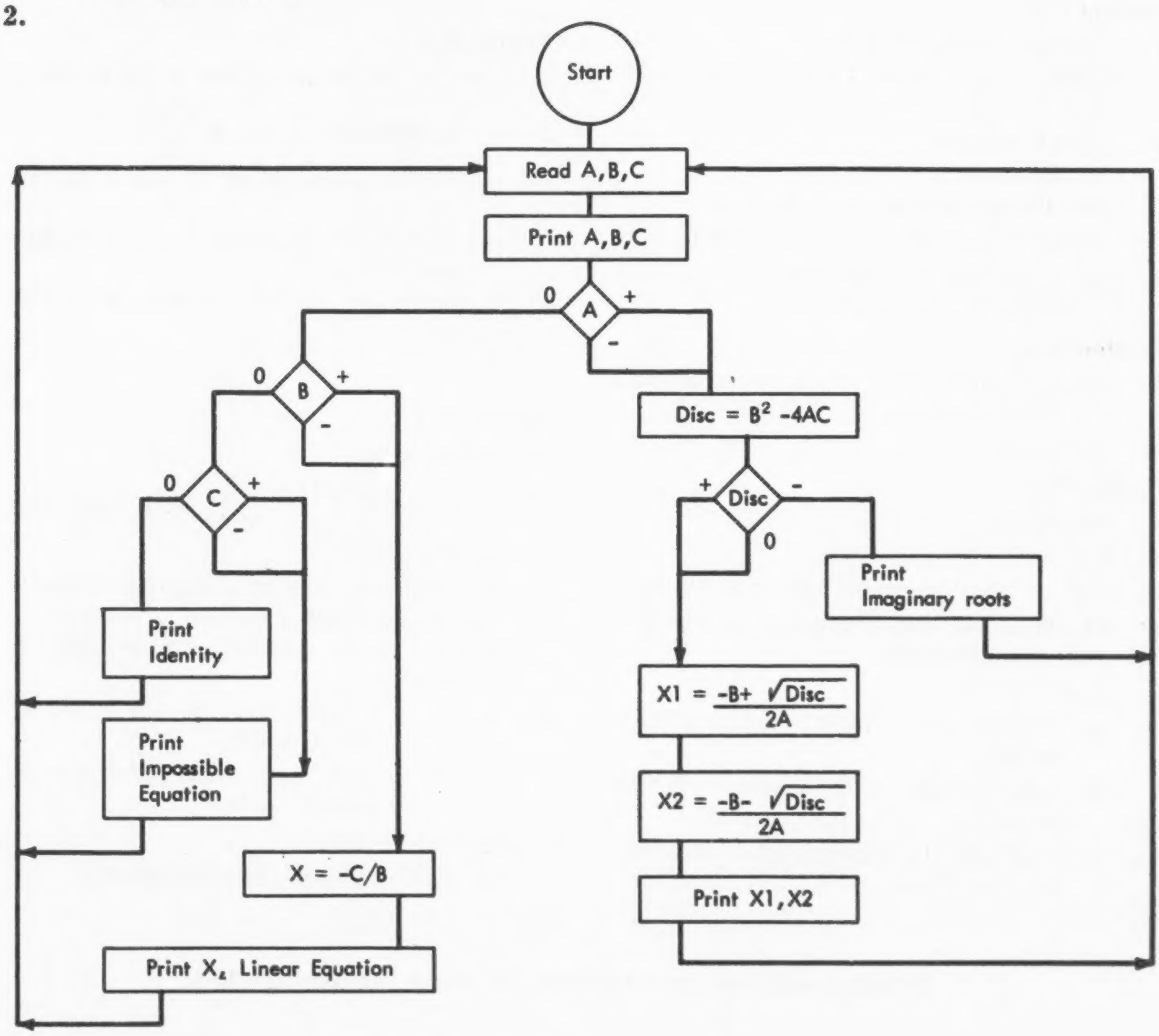
Chapter 6 —COMPUTERS ARE NEEDED

6–1 Computers are essential

6–2 Flow charts

1. Yes
   (a) Line segment connecting points $(0, -B)$ and $(0, B)$.
   (b) The point $(0, 0)$.

2.


Start
Read A,B,C
Print A,B,C
0
A
+
-
0
B
+
-
0
C
+
-
Disc = B² -4AC
+
Disc
-
0
Print Imaginary roots
Print Identity
Print Impossible Equation
X = -C/B
Print X, Linear Equation
X1 = -B+ √Disc / 2A
X2 = -B- √Disc / 2A
Print X1,X2

6–3 Graph of an ellipse

1. Program:

```
C       ROOTS OF QUADRATIC EQUATION
    1   READ,A,B,C
        PRINT 2,A,B,C
    2   FORMAT(/3F8.2)
        IF (A)3,4,3
    4   IF (B)5,6,5
    5   X=(-1.0*C)/B
        PRINT 7,X
    7   FORMAT(F8.2,19H     LINEAR EQUATION)
        GO TO 1
    6   IF (C)8,9,8
    9   PRINT 10
   10   FORMAT(12H     IDENTITY)
        GO TO 1
    8   PRINT 11
   11   FORMAT(23H     IMPOSSIBLE EQUATION)
        GO TO 1
    3   DISC=B*B-4.0*A*C
        IF (DISC)12,13,13
   12   PRINT 14
   14   FORMAT(19H     IMAGINARY ROOTS)
        GO TO 1
   13   X1=(-1.0*B+SQRT(DISC))/(2.0*A)
        X2=(-1.0*B-SQRT(DISC))/(2.0*A)
        PRINT 15,X1,X2
   15   FORMAT(2F8.2)
        GO TO 1
        END
 1.0, -4.0, 4.0
 2.0, -7.0, 3.0
 0.0, 1.0, 4.0
 0.0, 0.0, 0.0
 0.0, 0.0, 1.0
 1.0, 1.0, 8.0
 1.0, 5.0, 3.0
```

Output

```
    1.00   -4.00    4.00    2.00    2.00
    2.00   -7.00    5.00    3.00     .50
     .00    1.00    4.00   -4.00   LINEAR EQUATION
     .00     .00     .00   IDENTITY
     .00     .00    1.00   IMPOSSIBLE EQUATION
    1.00    1.00    8.00   IMAGINARY ROOTS
    1.00    5.00    3.00    -.65   -4.30
```

2. Program:

```
C       GOTRAN PROGRAM TO PLOT GRAPH OF ELLIPSE
C       1ST AND 2ND QUADRANT OF Y2 + 9X2 = 2304
        X=-16.0
    1   X1=X*X
        X2=9.0*X1
        X3=2304.0-X2
        Y=SQR(X3)
        PRINT,X,Y
        PLOT(Y,*)
        X=X+1.0
        X4=16.0-X
        IF (X4)2,1,1
    2   STOP
        END
```

Output:

```
-16.000000    0.0             *
-15.000000    16.703293                       *
-14.000000    23.237900                              *
-13.000000    27.982135                                  *
-12.000000    31.749014                                      *
-11.000000    34.856850                                         *
-10.000000    37.469987                                            *
-9.0000000    39.686270                                              *
-8.0000000    41.569220                                                *
-7.0000000    43.162484                                                  *
-6.0000000    44.497192                                                   *
-5.0000000    45.596850                                                    *
-4.0000000    46.475801                                                     *
-3.0000000    47.148703                                                      *
-2.0000000    47.623522                                                      *
-1.0000000    47.906160                                                      *
  0.0         47.999999                                                      *
 1.0000000    47.906160                                                      *
 2.0000000    47.623522                                                      *
 3.0000000    47.148703                                                      *
 4.0000000    46.475801                                                     *
 5.0000000    45.596850                                                    *
 6.0000000    44.497192                                                   *
 7.0000000    43.162484                                                  *
 8.0000000    41.569220                                                *
 9.0000000    39.686270                                              *
 10.000000    37.469987                                            *
 11.000000    34.856850                                         *
 12.000000    31.749014                                      *
 13.000000    27.982135                                  *
 14.000000    23.237900                              *
 15.000000    16.703293                       *
 16.000000    0.0             *
STOP          END OF PROGRAM
```

3. Program:

```
C       PERCENT OF EARTH VISIBLE FROM SATELLITE
        H=200.0
    1   N=H
        R=4000.0+H
        S=SQRT(H*(H+8000.0))
        A=ATAN(S/4000.0)
        B=4000.0*COS(A)
        HZ=4000.0-B
        AREA=25132.742*HZ
        PERC=HZ/8000.0
        PRINT 2,N,AREA,PERC
    2   FORMAT(/I5,F14.2,F8.4)
        H=H+200.0
        IF (H-5000.0)1,1,3
    3   STOP
        END
```

Output:

```
Elevation      Area        Decimal part
                             of area
   200      4787189.20       .0238
   400      9139177.30       .0454
   600     13112737.00       .0652
   800     16755160.00       .0833
  1000     20106191.00       .0999
  1200     23199456.00       .1153
  1400     26063583.00       .1296
  1600     28723135.00       .1428
  1800     31199268.00       .1551
  2000     33510324.00       .1666
  2200     35672273.00       .1774
  2400     37699113.00       .1875
  2600     39603114.00       .1969
  2800     41395111.00       .2058
  3000     43084705.00       .2142
  3200     44680433.00       .2222
  3400     46189908.00       .2297
  3600     47619934.00       .2368
  3800     48976622.00       .2435
  4000     50265479.00       .2499
  4200     51491469.00       .2560
  4400     52659081.00       .2619
  4600     53772381.00       .2674
  4800     54835074.00       .2727
  5000     55850542.00       .2777
 STOP
```

4. Program:

```
C     PERCENT OF EARTH VISIBLE  26000 MILE ELEVATION
      H=200.0
    1 R=4000.0+H
      R=4000.0+H
      S=SQRT(H*(H+8000.0))
      A=ATAN(S/4000.0)
      B=4000.0*COS(A)
      HZ=4000.0-B
      AREA=25132.742*HZ
      PERC=HZ/8000.0
      PUNCH 2,H,AREA,PERC
    2 FORMAT(F8.0,F14.2,F8.4)
      H=H+200.0
      IF (H-5000.0)1,1,3
    3 H=H+800.0
      IF (H-25000.0)1,1,4
    4 STOP
      END
```

Output:

```
Elevation      Area        Decimal part
                             of area
   200.     4787189.20       .0238
   400.     9139177.30       .0454
   600.    13112737.00       .0652
   800.    16755160.00       .0833
  1000.    20106191.00       .0999
  1200.    23199456.00       .1153
  1400.    26063583.00       .1296
  1600.    28723135.00       .1428
  1800.    31199268.00       .1551
  2000.    33510324.00       .1666
  2200.    35672273.00       .1774
  2400.    37699113.00       .1875
  2600.    39603114.00       .1969
  2800.    41395111.00       .2058
  3000.    43084705.00       .2142
  3200.    44680433.00       .2222
  3400.    46189908.00       .2297
  3600.    47619934.00       .2368
  3800.    48976622.00       .2435
  4000.    50265479.00       .2499
  4200.    51491469.00       .2560
  4400.    52659081.00       .2619
  4600.    53772381.00       .2674
  4800.    54835074.00       .2727
  5000.    55850542.00       .2777
  6000.    60318881.00       .3000
  7000.    63974254.00       .3181
  8000.    67020644.00       .3333
  9000.    69598363.00       .3461
 10000.    71807833.00       .3571
 11000.    73722707.00       .3666
 12000.    75398231.00       .3750
 13000.    76876819.00       .3823
 14000.    78190750.00       .3888
 15000.    79365853.00       .3947
 16000.    80424774.00       .4000
 17000.    81382211.00       .4047
 18000.    82252613.00       .4090
 19000.    83047388.00       .4130
 20000.    83775813.00       .4166
 21000.    84446018.00       .4200
 22000.    85064673.00       .4230
 23000.    85637499.00       .4259
 24000.    86169405.00       .4285
 25000.    86664634.00       .4310
```

6–4 Area under curve

1. Program:

```
C        AREA UNDER ELLIPSE X2 + 4Y2 = 16
       1 READ,N
         X=0.0
         SUM=0.0
         CX=N
         DO 2 K=1,N
         SUM=SUM+0.5*SQRT(16.0-X*X)
       2 X=X+4.0/CX
         ABOVE=4.0/CX*SUM
         UNDER=4.0/CX*(SUM-8.0/CX)
         AREA=(ABOVE+UNDER)*2.0
         PRINT 3,N,AREA
       3 FORMAT(/I6,F10.6)
         GO TO 1
         END
4
8
25
50
100
```

Output:

```
     4 23.965672
     8 25.718550
    25 25.595166
    50 25.400542
   100 25.276890
```

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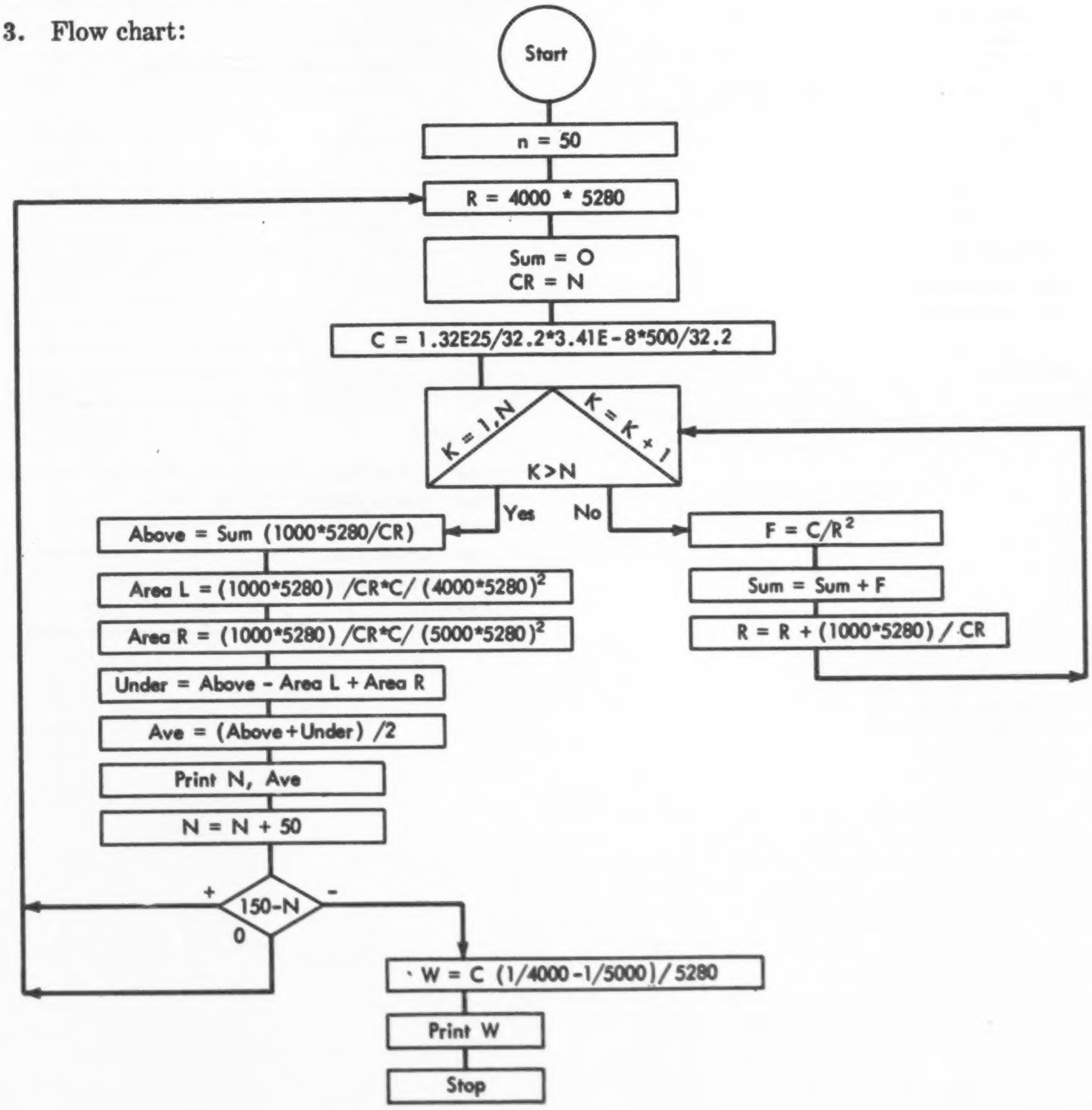
2. Program:

```
C      AREA BETWEEN 4X=Y2 AND 4Y=X2
     1 READ.N
       X=0.0
       SUM1=0.0
       SUM2=0.0
       CX=N
       DO 2 K=1.N
       Y1=2.0*SQRT(X)
       Y2=X*X/4.0
       SUM1=SUM1+Y1
       SUM2=SUM2+Y2
     2 X=X+4.0/CX
       UND1=SUM1*4.0/CX
       UND2=SUM2*4.0/CX
       ABV1=(4.0+SUM1)*4.0/CX
       ABV2=(4.0+SUM2)*4.0/CX
       AREA=(UND1+ABV1)/2.0-(UND2+ABV2)/2.0
       PRINT 3.N.AREA
     3 FORMAT(/I6.F10.5)
       GO TO 1
       END
   4
   8
   25
   50
   100
```

Output:

```
     4   4.79252
     8   5.15508
    25   5.30352
    50   5.32312
   100   5.32979
```

3. Flow chart:


Start
n = 50
R = 4000 * 5280
Sum = O
CR = N
C = 1.32E25/32.2*3.41E-8*500/32.2
K = 1,N
K = K + 1
K>N
Yes
No
Above = Sum (1000*5280/CR)
Area L = (1000*5280) /CR*C/ (4000*5280)²
Area R = (1000*5280) /CR*C/ (5000*5280)²
Under = Above - Area L + Area R
Ave = (Above+Under) /2
Print N, Ave
N = N + 50
F = C/R²
Sum = Sum + F
R = R + (1000*5280) / CR
+
150-N
-
0
W = C (1/4000-1/5000)/5280
Print W
Stop

Program:

```
C       WORK DONE TO GET SCOUT ROCKET TO 1000 MILE ELEVATION
        N=50
      1 R=4000.0*5280.0
        SUM=0.0
        CR=N
        C=1.32E25/32.2*3.41E- 8*500.0/32.2
        DO 2 K=1,N
        F=C/(R*R)
        SUM=SUM+F
      2 R=R+5.28E 6/CR
        ABOVE=5.28E 6/CR*SUM
        UNDER=ABOVE-(C/(CR*5.28E 6)*(1.0/16.0-1.0/25.0))
        AVE=(ABOVE+UNDER)/2.0
        PRINT 3,N,AVE
      3 FORMAT(/I7,E16.0)
        N=N+50
        IF (150-N)4,1,1
      4 W=C/5.28E 6*(1.0/4.0-1.0/5.0)
        PRINT 5,W
      5 FORMAT(/E16.0)
        STOP
        END
```

Output:

```
     50   20555435.E+02
    100   20555274.E+02
    150   20555243.E+02
   20555244.E+02
STOP
```

4. Program:

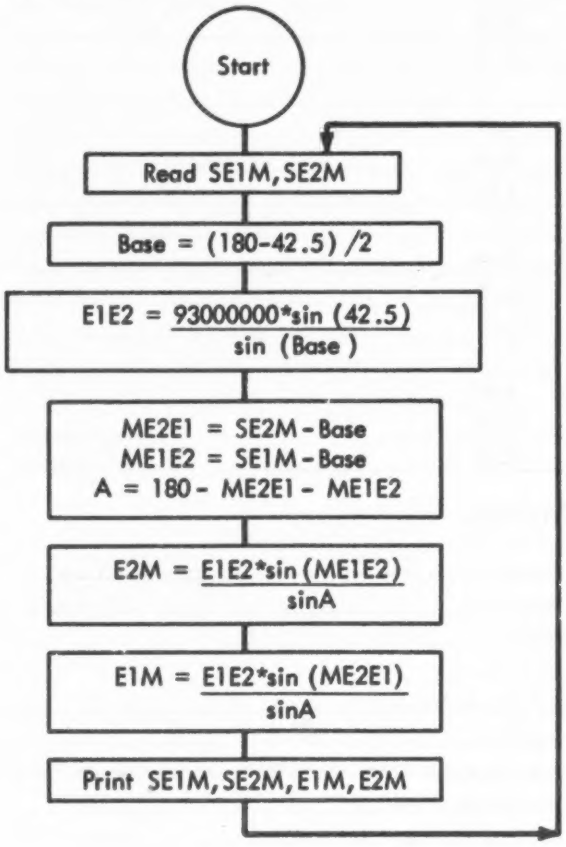
```
C       WORK DONE TO GET SPACECRAFT TO A GIVEN ELEVATION
      1 READ,R,RL,WGHT
        R=R*5280.0
        RL=RL*5280.0
        N=50
        RN=R
      2 R=RN
        SUM=0.0
        CR=N
        DIFF=RL-R
        C=1.32E25/32.2*3.41E- 8*WGHT/32.2
        DO 3 K=1,N
        F=C/(R*R)
        SUM=SUM+F
      3 R=R+DIFF/CR
        AROVF=DIFF/CR*SUM
        UNDER=ABOVE-DIFF/CR*(C/(RN*RN)-C/(RL*RL))
        AREA=(ABOVE+UNDER)/2.0
        PRINT 6,N,AREA
      6 FORMAT(/I8,E16.0)
        N=N+50
        IF (150-N)4,2,2
      4 W=C*(1.0/RN-1.0/RL)
        PRINT 5,W
      5 FORMAT(/E16.0)
        GO TO 1
        END
4000.0, 5000.0, 500.0
4000.0, 216000.0, 90000
```

Output:

```
     50   21131455.E+05
    100   18981792.E+05
    150   18533133.E+05
    200   18370696.E+05
    250   18294446.E+05
  18157133.E+05
```

6–5 The distance between Earth and Mars

1. Flow chart


Start
Read SE1M, SE2M
Base = (180-42.5) /2
E1E2 = 93000000*sin (42.5) / sin (Base)
ME2E1 = SE2M - Base
ME1E2 = SE1M - Base
A = 180 - ME2E1 - ME1E2
E2M = E1E2*sin (ME1E2) / sinA
E1M = E1E2*sin (ME2E1) / sinA
Print SE1M, SE2M, E1M, E2M

Program:

```
C       DISTANCE BETWEEN EARTH AND MARS
      1 READ,SE1M,SE2M
        SE1M=SE1M*3.1415927/180.0
        SE2M=SE2M*3.1415927/180.0
        BASE=(180.0-42.5)*3.1415927/360.0
        E1E2=93000000.0*SIN(42.5*3.1415927/180.0)/SIN(BASE)
        A=3.1415927-SE1M-SE2M+2.0*BASE
        E2M=E1E2*SIN(SE1M-BASE)/SIN(A)
        E1M=E1E2*SIN(SE2M-BASE)/SIN(A)
        PRINT 2,SE1M,SE2M,E1M,E2M
      2 FORMAT(/2F8.4,2E12.0)
        GO TO 1
        END
118.0, 148.0
140.0, 110.0
106.0, 155.0
```

Output:

```
2.0594  2.5830 846277.E+02 652562.E+02
2.4434  1.9198 481110.E+02 690954.E+02
1.8500  2.7052 806694.E+02 489335.E+02
```

2. Program:

```
C       HEIGHT OF MODEL ROCKET BY TWO OBSERVERS
        DIMENSION X(4)
      1 DO 2 K=1,4
        READ,X(K)
      2 X(K)=X(K)*3.1415927/180.0
        C=3.1415927-X(1)-X(3)
        BA=1000.0*SIN(X(3))*SIN(X(2))/(SIN(C)*COS(X(2)))
        BB=1000.0*SIN(X(1))*SIN(X(4))/(SIN(C)*COS(X(4)))
        AVE=(BA+BB)/2.0
        DIFF=ABS(BA-BB)
        PERC=DIFF/AVE
        IF (PERC-0.1)3,3,5
      3 PRINT 4,BA,BB
      4 FORMAT(/2F10.2)
        GO TO 1
      5 PRINT 6,PERC
      6 FORMAT(/24H INACCURATE OBSERVATIONS,F8.4)
        GO TO 1
        END
30.0, 30.0, 60.0, 45.0
40.0, 40.0, 65.0, 50.0
37.0, 48.0, 68.0, 60.0
52.0, 31.0, 70.0, 32.0
12.0, 56.0, 104.0, 81.0
64.0, 73.0, 43.0, 68.0
```

Output:

```
   499.99    500.00
   787.30    793.06
  1066.06   1079.14
INACCURATE OBSERVATIONS   .1366
  1600.50   1460.51
  2332.64   2326.23
```

3. Program:

```
C       DATA FOR TEN STARS
      1 READ,PC,APM,TEMP
        DLY=PC*3.26
        DAU=DLY*63000.0
        P=1.0/PC
        IF (P-0.01)3,5,5
      3 PRINT 4
      4 FORMAT(/33H   ERROR GREATER THAN 50 PER CENT)
      5 ABM=APM+5.0-5.0*(LOG(PC)/2.3025851)
        B=2.5**(4.9-ABM)
        A=(5500.0/TEMP)**4
        R=SQRT(A*B*432000.0*432000.0)
        PRINT 6,DLY,DAU,P,ABM,B,R
      6 FORMAT(/F8.1,F13.2,F12.2,F8.1,F8.1,F12.0)
        GO TO 1
        END
1.32, 0.3, 5800.0
2.66, -1.44, 10000.0
3.47, 0.38, 6500.0
5.05, 0.76, 8000.0
8.14, 0.001, 10700.0
0.0000048, -26.7, 5500.0
11.1, -0.06, 4000.0
13.7, 0.9, 5200.0
20.8, 0.78, 3600.0
25.6, 1.33, 13000.0
```

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Output:

```
 DLY    DAU              P      ABM   B      R
  4.3   271101.60       .75     4.6   1.2    426302.
  8.6   546310.80       .37     1.4   23.9   639031.
 11.3   712668.60       .28     2.6   7.7    863779.
 16.4  1037169.00       .19     2.2   11.4   689579.
 26.5  1671793.20       .12     .4    59.1   877566.
   .0         .98  208333.33    4.8   1.0    433230.
 36.1  2279718.00       .09    -.2   115.8   8791639.
 44.6  2813706.00       .07     .2    73.0   4131415.
 67.8  4271904.00       .04    -.8   187.2  13797033.
 83.4  5257728.00       .03    -.7   170.9   1011073.
```

6–6 Circumference of an ellipse

1. Program:

```
C     LENGTH OF CURVE  Y = X3 - 9X2 + 24X
    4 READ,X,XR
       PRINT 1,X,XR
    1 FORMAT(/2F8.2)
      XX=X
      N=10
      DIFF=XR-X
    5 SUM=0.0
      CX=N
      BASE2=DIFF/CX*DIFF/CX
      YL=X*(X*(X-9.0)+24.0)
      DO 2 K=1,N
      X=X+DIFF/CX
      Y=X*(X*(X-9.0)+24.0)
      HYP=SQRT(BASE2+(Y-YL)*(Y-YL))
      SUM=SUM+HYP
    2 YL=Y
      PRINT 3,N,SUM
    3 FORMAT(/I6,F10.2)
      X=XX
      N=N+90
      IF (100-N)4,5,5
      END
0.0, 5.0
1.0, 4.0
0.5, 8.5
-1.0, 5.0
```

Output:

```
   .00     5.00
    10    28.90
   100    29.06

  1.00     4.00
    10     8.75
   100     8.81

   .50     8.50
    10   166.48
   100   167.11

 -1.00     5.00
    10    62.76
   100    63.08
```

2. Program:

```
C     APPROXIMATION FORMULAS FOR CIRCUMFERENCE OF ELLIPSE
      A=30.0
      B=2.0
      DO 2 K=1,15
      E=SQRT(A*A-B*B)/A
      E=E*E
      C=6.2831853*A*(1.-E*(.25+E*(3./64.+E*(5./256.+175.*E/16384.))))
      C1=3.1415927*(A+B)
      C2=3.1415927*(1.5*(A+B)-SQRT(A*B))
      C3=3.1415927*SQRT(2.0*(A*A+B*B))
      C4=(C1+C3)/2.0
      PUNCH 3,A,B,C,C1,C2,C3,C4
    3 FORMAT(7F9.1)
    2 B=B+2.0
      STOP
      END
```

Output:

```
   A        B        C        C1          C2          C3          C4
 30.0      2.0   [illegible] [illegible] [illegible] [illegible] [illegible]
 30.0      4.0   [illegible] [illegible] [illegible] [illegible] [illegible]
 30.0      6.0   [illegible] [illegible] [illegible] [illegible] [illegible]
 30.0      8.0   [illegible] [illegible] [illegible] [illegible] [illegible]
 30.0     10.0   [illegible] [illegible] [illegible] [illegible] [illegible]
 30.0     12.0   [illegible] [illegible] [illegible] [illegible] [illegible]
 30.0     14.0   [illegible] [illegible] [illegible] [illegible] [illegible]
 30.0     16.0   [illegible] [illegible] [illegible] [illegible] [illegible]
 30.0     18.0   [illegible] [illegible] [illegible] [illegible] [illegible]
 30.0     20.0   [illegible] [illegible] [illegible] [illegible] [illegible]
 30.0     22.0   [illegible] [illegible] [illegible] [illegible] [illegible]
 30.0     24.0   [illegible] [illegible] [illegible] [illegible] [illegible]
 30.0     26.0   [illegible] [illegible] [illegible] [illegible] [illegible]
 30.0     28.0   [illegible] [illegible] [illegible] [illegible] [illegible]
 30.0     30.0   [illegible] [illegible] [illegible] [illegible] [illegible]
```

3. **Program:**

```
C       CIRCUMFERENCE OF ELLIPSE USING FORMULAS GIVEN RA AND RP
    1   READ,RA,RP
        A=(RA+RP)/2.0
        B2=RA*RP
        B=SQRT(B2)
        E=SQRT(A*A-B*B)/A
        E=E*E
        C=6.2831853*A*(1.-E*(.25+E*(3./64.+E*(5./256.+175.*E/16384.))))
        C1=3.1415927*(A+B)
        C2=3.1415927*(1.5*(A+B)-SQRT(A*B))
        C3=3.1415927*SQRT(2.0*(A*A+B*B))
        C4=(C1+C3)/2.0
        PRINT 2,RA,RP,C,C1,C2,C3,C4
    2   FORMAT(/7F9.1)
        GO TO 1
        END
  7000.0, 4900.0
  4680.0, 4342.0
  5048.0, 4948.0
  5423.0, 4258.0
  4370.0, 4344.0
  4442.0, 4425.0
```

Output:

```
   RA       RP       C        C1       C2       C3       C4
 7000.0   4900.0  37092.1  37091.5  37092.1  37092.7  37092.1
 4680.0   4342.0  28333.5  28333.5  28333.5  28333.5  28333.5
 5048.0   4948.0  31402.5  31402.5  31402.5  31402.5  31402.5
 5423.0   4258.0  30303.3  30303.2  30303.3  30303.4  30303.3
 4370.0   4344.0  27375.7  27375.7  27375.7  27375.7  27375.7
 4442.0   4425.0  27856.4  27856.4  27856.4  27856.4  27856.4
```

6–7 Velocity along an elliptical arc

1. VP ≈ 17300 mph.
   VA ≈ 16100 mph.
2. T ≈ 105 minutes.
3. V ≈ 1610 mph.
   R ≈ 4790 miles.
   T ≈ 112 minutes.
4. **Program:**

```
C       GOTRAN PROGRAM--ECCENTRICITY OF ELLIPSE GIVEN RA AND RP
    1   READ,RA,RP
        R1=RA+RP
        R=R1/2.0
        E1=R-RP
        E=E1/R
        PRINT,RA,RP,E
        GO TO 1
        END
  7000.0
  4900.0
  4680.0
  4342.0
  5048.0
  4948.0
  5423.0
  4258.0
  4370.0
  4344.0
  4442.0
  4425.0
```

Output:

```
7000.0000        4900.0000        .17647058
4680.0000        4342.0000        3.7463976E-02
5048.0000        4948.0000        1.0004001E-02
5423.0000        4258.0000        .12033880
4370.0000        4344.0000        2.9837043E-03
4442.0000        4425.0000        1.9172211E-03
```

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5. Program:

```
C     FINDING VELOCITY AND TIME OF ORBIT GIVEN APOGEE AND PERIGEE
    1 READ,A,P
      RA=4000.0+A
      RP=4000.0+P
      R=(RA+RP)/2.0
      V=1115000.0/SQRT(R)
      VA=V*SQRT(RP/RA)
      VP=V*SQRT(RA/RP)
      T=0.000338*(R**1.5)
      PRINT 2,RA,RP,VA,VP,V,T
    2 FORMAT(/6F9.1)
      GO TO 1
      END
21650.0, 21400.0
510.0, 457.0
468.0, 436.0
22900.0, 22200.0
634.0, 540.0
358.0, 346.0
```

Output:

```
    RA        RP        VA       VP        V         T
 25650.0  25400.0    6944.8   7013.2   6978.9   1370.3
  4510.0   4457.0  16553.8  16750.7  16651.9    101.4
  4468.0   4436.0  16650.8  16770.9  16710.8    100.4
 26900.0  26200.0    6753.3   6933.7   6842.9   1462.2
  4634.0   4540.0  16295.2  16632.6  16463.0    105.0
  4350.0   4346.0  16878.4  16925.0  16901.7     97.0
```

6. Program:

```
C     FINDING TIME VELOCITY DATA GIVEN PERIGEE AND VELOCITY AT PERIGEE
    1 READ,P,VP
      RP=4000.0+P
      RA=(RP*RP*VP*VP)/(2.0*1115000.0*1115000.0-VP*VP*RP)
      R=(RA+RP)/2.0
      V=1115000.0/SQRT(R)
      VA=V*SQRT(RP/RA)
      T=0.000338*(R**1.5)
      PRINT 2,RA,RP,VA,VP,V,T
    2 FORMAT(/6F9.1)
      GO TO 1
      END
22160.0, 6870.0
642.0, 16540.0
425.0, 16790.0
160.0, 17290.0
```

Output:

```
    RA        RP        VA       VP        V         T
 25802.4  26160.0    6965.1   6870.0   6917.4   1415.4
  4845.7   4642.0  15844.6  16540.0  16188.5    110.4
  4455.0   4425.0  16676.9  16790.0  16733.3     99.9
  4162.5   4160.0  17279.3  17290.0  17284.6     90.7
```

6–8 Curve fitting

1. Program:

```
C     LINEAR CURVE FITTING BY METHOD OF AVERAGE POINTS
      DIMENSION X(30),Y(30)
    9 N=0
      SUMX1=0.0
      SUMY1=0.0
      SUMX2=0.0
      SUMY2=0.0
    1 N=N+1
      READ,X(N),Y(N)
      NUM=N
      IF (SENSE SWITCH 9)2,1
    2 I=2
      N1=N-1
      DO 3 K=1,N1
      DO 4 J=I,N
      IF (X(K)-X(J))4,4,5
    5 SAVE=X(K)
      X(K)=X(J)
      X(J)=SAVE
      SAVE=Y(K)
      Y(K)=Y(J)
      Y(J)=SAVE
    4 CONTINUE
    3 I=I+1
      M=NUM/2
      M1=M
      DO 6 K=1,M
      SUMX1=SUMX1+X(K)
    6 SUMY1=SUMY1+Y(K)
      C1=M
      AVEX1=SUMX1/C1
      AVEY1=SUMY1/C1
      M1=M1+1
      L=NUM-M
      DO 7 K=M1,NUM
      SUMX2=SUMX2+X(K)
    7 SUMY2=SUMY2+Y(K)
      C1=L
      AVEX2=SUMX2/C1
      AVEY2=SUMY2/C1
      COEFX=(AVEY1-AVEY2)/(AVEX1-AVEX2)
      B=AVEY1-COEFX*AVEX1
      PRINT 8,COEFX,B
    8 FORMAT(/2F10.4)
      GO TO 9
      END
-5.2, 11.33
2.9, -10.55
-0.4, 4.1
-5.05, 11.33
1.1, -5.2
-1.11, 4.73
-0.71, 2.2
-0.7, 10.35
2.38, -8.75
-2.4, 7.37
2.45, -6.9
-2.4, 7.5
0.15, 0.3
-4.88, 9.3
```

Output:

```
COEFX      B
-2.3763  .2964
```

2. Program:

```
C      LINEAR CURVE FITTING BY METHOD OF LEAST SQUARES
       DIMENSION X(30),Y(30)
     4 SUMX=0.0
       SUMY=0.0
       SUMXY=0.0
       SUMX2=0.0
       C=0.0
       N=0
     1 N=N+1
       READ,X(N),Y(N)
       C=C+1.0
       SUMX=SUMX+X(N)
       SUMY=SUMY+Y(N)
       SUMX2=SUMX2+X(N)*X(N)
       SUMXY=SUMXY+X(N)*Y(N)
       IF (SENSE SWITCH 9)2,1
     2 DEN=SUMX2*C-SUMX*SUMX
       B=(SUMX2*SUMY-SUMX*SUMXY)/DEN
       COEFX=(SUMXY*C-SUMX*SUMY)/DEN
       PRINT 3,B,COEFX
     3 FORMAT(/2F10.4)
       GO TO 4
       END
-9.2, 11.33
2.9, -10.55
-0.4, 4.1
-5.05, 11.33
1.1, -9.2
-1.11, 4.73
-0.71, 2.2
-0.7, 10.35
2.30, -0.75
-2.4, 7.37
2.45, -6.9
-2.4, 7.5
0.15, 0.3
-4.88, 9.3
```

Output:

```
   B      COEFX
 .1176   -2.5568
```

3.
$$
\begin{aligned}
A\Sigma X^4 + B\Sigma X^3 + C\Sigma X^2 &= \Sigma X^2Y\\
A\Sigma X^3 + B\Sigma X^2 + C\Sigma X &= \Sigma XY\\
A\Sigma X^2 + B\Sigma X + CN &= \Sigma Y
\end{aligned}
$$

4. Program:

```
C      CURVE FITTING FOR THE PARABOLA
       DIMENSION X(30),Y(30)
     4 SX4=0.0
       SX3=0.0
       SX2=0.0
       SX=0.0
       SXY=0.0
       SY=0.0
       SX2Y=0.0
       CNT=0.0
       N=0
     1 N=N+1
       READ,X(N),Y(N)
       CNT=CNT+1.0
       SX=SX+X(N)
       SY=SY+Y(N)
       SXY=SXY+X(N)*Y(N)
       SX2=SX2+X(N)*X(N)
       SX3=SX3+X(N)**3
       SX4=SX4+X(N)**4
       SX2Y=SX2Y+X(N)*X(N)*Y(N)
       IF (SENSE SWITCH 9)2,1
     2 D1=(SX4*SX2*CNT)+(2.0*SX3*SX2*SX)
       D2=(SX2**3)+(SX*SX*SX4)+(SX3*SX3*CNT)
       A1=(SX2Y*SX2*CNT)+(SX*SX2*SXY)+(SX*SY*SX3)
       A2=(SX2*SX2*SY)+(SX*SX*SX2Y)+(SXY*SX3*CNT)
       B1=(SX4*SXY*CNT)+(SX3*SX2*SY)+(SX2Y*SX*SX2)
       B2=(SX2*SX2*SXY)+(SX*SY*SX4)+(SX3*SX2Y*CNT)
       C1=(SX4*SX2*SY)+(SX3*SX*SX2Y)+(SX3*SXY*SX2)
       C2=(SX2*SX2*SX2Y)+(SX3*SX3*SY)+(SX4*SX*SXY)
       A=(A1-A2)/(D1-D2)
       B=(B1-B2)/(D1-D2)
       C=(C1-C2)/(D1-D2)
       PRINT 3,A,B,C
     3 FORMAT(/3F10.4)
       GO TO 4
       END
-2.0, 1.9
-1.4, 0.3
-1.0, 0.0
-0.4, -0.3
-0.2, -0.1
0.4, -0.1
1.1, 0.5
1.6, 0.9
2.1, 1.9
2.4, 2.3
```

Output:

```
  A       B       C
 .4682   .0188   -.2682
```

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